Effective Field Theory and Applications

有效场论及应用

Weak Field Observables from Scattering Amplitudes in Quantum Field Theory

量子场论散射振幅得出的弱场可观测量

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摘要

In this chapter, we will review the field-theoretic treatment of general relativity based on an effective field theory extension of the Einstein-Hilbert action. This pragmatic route to low-energy quantum effects in gravity critically underpins miscellaneous investigations of phenomenological and quantum extensions of general relativity. We discuss how it allows quantum field theory to be a theoretical laboratory for testing Einstein's theory of gravity. We also demonstrate the current state of an efficient and practical scheme for evaluating the classical components of perturbative weak field scattering amplitudes until the fourth post-Minkowskian order. Such results complement numerical predictions in Einstein's theory of gravity.

本章我们将回顾基于爱因斯坦-希尔伯特作用量有效场论推广的广义相对论场论处理方法。这种处理引力低能量子效应的实用方法，为广义相对论的各类唯象推广与量子推广研究提供了关键基础。我们将讨论它如何使量子场论成为检验爱因斯坦引力理论的理论实验室。我们还将展示目前一套高效实用方案的发展现状，该方案可用于计算微扰弱场散射振幅的经典分量，直至后闵可夫斯基四阶。这些结果是对爱因斯坦引力理论数值预言的补充。

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Keywords

关键词

Quantum field theories of gravity - Effective field theory - Classical Einstein gravity - Gravity phenomenology - Observables in general relativity

引力量子场论-有效场论-经典爱因斯坦引力-引力现象学-广义相对论中的可观测量

Introduction

引言

The present cosmological paradigm hinges on general relativity - Einstein's theory of gravity - crafted as a relativistic theory of curved space-time. Gravity is inferred universal - with dynamics that couple equally to all varieties of matter. This framework successfully conditions our image of the shape and dynamics of the universe. We introduce dark energy and matter because general relativity fails to explain the observed dynamics and stability of the galaxies or the accelerated cosmological expansion, and the tension in the measured value for the acceleration rate [1] (For example, from local astrophysical data and the cosmic microwave background and anomalies in the current cosmological model [2].), bids us to assess transformations of the law of gravity over large astrophysical scales.

当前宇宙学范式以广义相对论 (即爱因斯坦的引力理论) 为基础, 该理论是作为描述弯曲时空的相对论性理论构建的。引力被认为是普适的相互作用，其动力学同等耦合所有种类的物质。这一框架成功构建了我们对宇宙形态与动力学的认知。我们引入暗能量与暗物质，正是因为广义相对论无法解释观测到的星系动力学与稳定性，也无法解释宇宙的加速膨胀；而加速度测量值存在的张力 [1](例如，来自局部天体物理数据、宇宙微波背景的测量值与当前宇宙学模型的偏差 [2])，也要求我们重新评估引力定律在大天体物理尺度上的修正。

Recent cosmological surveys focus on understanding such queries, despite the weakness of the gravitational interactions that make direct detection of departures from Einstein's gravity very challenging. A new exciting development is observations of gravitational waves from Earth-based interferometers [3] that have lessened the observational gaps in the scales to which we gauge gravity with accuracy. This new pursuit aims to shed light on the exact gravitational attraction of massive astrophysical objects such as black holes and neutron stars. Planned space-based gravitational wave interferometers will even bring this a step further and have the capacity to address questions associated with gravitational attraction and cosmology.

尽管引力相互作用很弱，导致直接探测偏离爱因斯坦引力理论的偏差极具挑战性，但近期的宇宙学巡天仍聚焦于研究这类问题。一个令人振奋的新进展是，地基干涉仪观测到了引力波 [3]，缩小了我们能够精确测量引力的尺度范围的观测空白。这一新研究方向旨在阐明黑洞、中子星这类大质量天体物理物体的精确引力性质。规划中的天基引力波干涉仪将进一步推进这一领域，并有能力解答与引力和宇宙学相关的诸多问题。

We are thus constantly learning more about classical gravitational attraction, but quantum gravity is a critical puzzle in theoretical physics. Everywhere in nature, we encounter mechanics dynamics, yet a century after the creation of general relativity by Albert Einstein, a fundamental theory of quantum gravity still lacks an adequate resolution. In place of such a description of gravity - effective field theory offers a possible attractive avenue for exploring low-energy quantum gravity consequences.

因此，我们对经典引力的认知一直在不断拓展，但量子引力仍是理论物理学中一个关键谜题。力学动力学遍布自然界，然而在爱因斯坦提出广义相对论一个世纪之后，量子引力的基础理论仍未得到完善解答。在缺乏完备引力描述的现状下，有效场论为探索低能量子引力的效应提供了一条可行的优质路径。

The overarching focus of this chapter is to demonstrate how low-energy field theory methods provide a flexible and efficient starting point for quantum field theory applications in gravity. Held up against the geometrical framework of general relativity - this is a different way to think about gravity - a quantum field theory mediated by quanta - spin-2 gravitons - that propagate the field and generate metric and curvature in a perturbative expansion. As a field particle, the graviton is massless because, like photons, we give it an infinite range.

本章的核心目标是展示低能场论方法如何为量子场论在引力中的应用提供灵活高效的研究起点。和广义相对论的几何框架不同，这是一种思考引力的不同方式: 引力是由量子 (自旋为 2 的引力子) 传递的量子场论，引力子在微扰展开中传播场并产生度规与曲率。作为场粒子，引力子是无质量的，因此和光子一样，其作用范围是无穷远。

The advantage of this adaption of gravity is that it allows a direct unification with other fundamental forces at low energies in the context of the standard model. For instance, we verify the classical equivalence principle at the microscopic level by considering the scattering of different types of matter in the context of the bending of light around a huge massive star and demonstrate that the classical scattering angle is universal, as expected.

对引力做这种适配的优势在于，它允许引力在低能下直接和标准模型框架中的其他基本力统一。例如，我们可以通过研究大质量恒星偏折光背景下不同种类物质的散射，在微观层面验证经典等效原理，证明经典散射角正如预期那样是普适的。

Another point is improving the analysis of the measured gravitational wave signals in observations of binary in-spirals that will be capable of testing Einstein's classical theory [4]. Traditional perturbative (off-shell) quantum field theory calculations in gravity are far from optimal computation-wise. Direct computation from Feynman diagrams is notoriously complex and tiresome. It is far more problematic than for gluon scattering amplitudes in gauge theories as they rely on infinite vertices and endless index contractions. This dire need for progress evokes the search for inspiration in (on-shell) computational methods for particle physics amplitudes, which is recent, prompted by the requirements of the Large Hadron Collider advancement. Stimulated by early examples of gravity scattering amplitude computations [5-7] and [8,9], a dedicated program was proposed in [10,11], and this has been a catalyst for new technology. It exploits that classical physics materializes when quantum numbers are enormous. An excellent application is quantum scattering amplitudes with superheavy black holes as point particles (Fig. 1). For black holes with no spin, see, for instance, [12-25]. Current state of the art for such computations is [26-28] reaching fourth post-Minkowskian order, while the fifth post-Minkowskian order approached in the probe limit [29]. While our principal emphasis here will be the non-spinning amplitude-based computations, there are also significant improvements for spinning black holes (see, e.g., [30-48]) and some based on the world-line approaches, e.g., [48-54].

另一优势在于，它可以改进对双黑洞并合旋近观测中测得的引力波信号的分析，从而能够检验爱因斯坦的经典引力理论 [4]。传统引力的离壳微扰量子场论计算在计算效率上并不理想。从费曼图直接计算的复杂度与繁琐程度是公认的，它比规范理论中胶子散射振幅的计算问题更大，因为引力计算依赖无穷多顶点和无休止的指标收缩。这种对计算方法改进的迫切需求，推动人们从粒子物理振幅的新的在壳计算方法中寻找灵感，这类方法是近期大型强子对撞机的发展需求催生的。在早期引力散射振幅计算实例 [5-7] 和 [8,9] 的启发下，文献 [10,11] 提出了专门的研究计划，成为推动新计算技术发展的催化剂。该计划利用了经典物理出现在量子数极大时的性质，一个出色的应用是把超重黑洞当作点粒子的量子散射振幅计算 (图 1)。无自旋黑洞的相关研究例如可见 [12-25]。目前这类计算的最新进展已经达到后闵可夫斯基四阶 [26-28]，探针极限下的后闵可夫斯基五阶也已有相关研究 [29]。虽然本文我们主要聚焦基于振幅的无散射黑洞计算，但旋转黑洞的相关研究也已取得重大进展 (例如参见 [30-48])，还有部分研究基于世界线方法，例如 [48-54]。

A key point in such applications is a relationship that connects tree amplitudes for open and closed strings found by Kawai-Lewellen-Tye, a relationship dubbed as the ”double-copy” [55] or ”KLT” [56] (Fig. 2). The Yang-Mills theory and gravity are very different theories with rather different Lagrangian dynamics - but double-copy relations provide clear evidence for a new fundamental connection between these forces of nature. It implies that one can recycle amplitude techniques and provide a possible way to compute amplitude gravity directly from the Yang-Mills theory. It is highly non-trivial and with many intriguing implications that we are still grasping.

这类应用中的一个关键要点是川合-勒韦伦-泰 (Kawai-Lewellen-Tye) 发现的开弦与闭弦树图振幅之间的关系，该关系被称为“双拷贝”[55] 或“KLT 关系”[56](图 2)。杨-米尔斯理论和引力是差别极大的理论，拉格朗日动力学也非常不同，但双拷贝关系为这两种自然基本力之间的全新基础联系提供了明确证据。这意味着人们可以重复利用杨-米尔斯理论的振幅技术，提供了直接从杨-米尔斯理论计算引力振幅的可能路径。这一关系高度非平凡，拥有许多仍待我们探索的迷人推论。

We will undoubtedly continue to have endless questions regarding gravity and the evolution of the universe - we hope that the framework of the effective field theory extension of the Einstein-Hilbert action, combined with new efficient computational techniques, will inspire the formulation of a fundamental theory of gravity guided by observations.

关于引力和宇宙演化，我们无疑仍有无穷无尽的问题——我们希望，爱因斯坦-希尔伯特作用量的有效场论扩展框架，结合全新的高效计算技术，能够启发人们在观测的指引下构建引力的基础理论。

Fig. 1 Schematic inspiral of two black holes

图 1 双黑洞并合前旋进阶段示意图

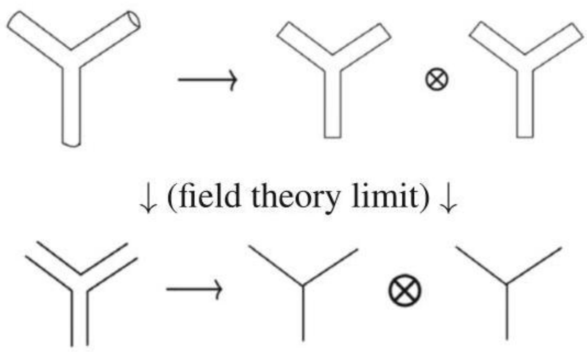


Fig. 2 Top: Schematic relation between a three-point correlation function for the closed string (left) and two open string correlation functions (right). Bottom: String theory induces a schematic relation between a gravity three-point vertex and two three-point gluon vertices

图 2 上图: 闭弦三点关联函数 (左) 与两个开弦关联函数 (右) 的关系示意图。下图: 弦理论导出的引力三点顶点与两个胶子三点顶点的关系示意图

We organize the chapter as follows. First, we will make a lightning review of the effective field theory treatment of general relativity. We will consider how treating general relativity as an effective field theory is a natural framework for investigations of gravitational attraction and how the effective field theory treatment splits the long-range infrared and short-distance ultraviolet physics contributions in terms of a well-defined energy scale expansion. In computations, this is helpful, as we can neglect short-distance physics compared to long-range physics at the length scales probed by observations. Next, we will introduce a framework for computations and consider how the reinstatement of $\hbar$ in momenta is needed for separating classical and quantum effects. We will compute gravitational scattering amplitudes using generalized unitarity and consider how to infer observables and metrics in general relativity from amplitudes. Finally, we will conclude and look ahead.

本章结构安排如下。首先，我们快速回顾广义相对论的有效场论处理方法。我们将探讨把广义相对论作为有效场论为何是研究引力相互作用的自然框架，以及有效场论处理如何按照定义清晰的能标展开，拆分长程红外物理和短程紫外物理的贡献。该方法在计算中十分有用，因为在观测可探测的长度尺度下，相较于长程物理我们可以忽略短程物理。接下来，我们介绍计算框架，说明为何需要在动量中恢复 $\hbar$ 以分离经典效应和量子效应。我们将利用广义幺正性计算引力散射振幅，并探讨如何从振幅推导出广义相对论中的可观测量和度规。最后，我们给出总结并对未来进行展望。

The following chapters appearing in this volume are related to the present discussion by presenting an effective field theory point of view from the compact binary system dynamics [57] and general discussion of quantum effects in general relativity from an effective field theory point of view [58].

本卷中的后续章节与本文讨论相关: 这些章节从紧致双星系统动力学的角度给出了有效场论观点 [57]，并从有效场论视角对广义相对论中的量子效应进行了全面讨论 [58]。

Lightning Introduction of the Effective Field Theory Treatment of General Relativity

广义相对论有效场论处理方法简介

Treating general relativity as an effective field theory is a natural framework for explorations of gravitational attraction in the context of quantum field theory. Historically, it has been understood since the 1960s that Einstein's general theory based on the Einstein-Hilbert action relativity has a perturbation expansion in gravitational scattering amplitudes [59-62]. Starting from a path integral and with minimal assumptions, Feynman derived the Einstein-Hilbert action regarding gravity as a particle theory.

将广义相对论作为有效场论处理，是在量子场论框架下研究引力相互作用的自然体系。早在 20 世纪 60 年代人们就已经认识到，基于爱因斯坦-希尔伯特作用量的爱因斯坦广义相对论，在引力散射振幅中存在微扰展开 [59-62]。费恩曼从路径积分出发，在最少假设的前提下，将引力视为粒子理论推导出了爱因斯坦-希尔伯特作用量。

$$\mathcal{L} = \sqrt{-g}\left[\frac{R}{16\pi G_N}\right]. \tag{1}$$

It is customary to define a weak field expansion around the flat Minkowski spacetime metric $\eta_{\mu\upsilon}$ by $g_{\mu\upsilon}(x) \equiv \eta_{\mu\upsilon} + \sqrt{32\pi G_N} h_{\mu\upsilon}(x)$ and utilize this and the definition of the Riemann curvature tensor $R^{\mu}_{\upsilon\alpha\beta} \equiv \partial_\alpha \Gamma^{\mu}_{\upsilon\beta} - \partial_\beta \Gamma^{\mu}_{\upsilon\alpha} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\upsilon\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\upsilon\alpha}$, as well as the determinant of the metric field $\det(g_{\mu\upsilon})$, to expand the theory in powers of G_N [63]. Following a gauge-fixing procedure, one derives the following propagator in harmonic gauge as

通常的做法是对平直闵氏度规 $\eta_{\mu\upsilon}$ 做弱场展开 $g_{\mu\upsilon}(x) \equiv \eta_{\mu\upsilon} + \sqrt{32\pi G_N} h_{\mu\upsilon}(x)$，结合该展开、黎曼曲率张量 $R^{\mu}_{\upsilon\alpha\beta} \equiv \partial_\alpha \Gamma^{\mu}_{\upsilon\beta} - \partial_\beta \Gamma^{\mu}_{\upsilon\alpha} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\upsilon\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\upsilon\alpha}$ 的定义以及度规场的行列式 $\det(g_{\mu\upsilon})$，将理论按 G_N 的幂次展开 [63]。经过规范固定步骤，我们可以得到谐规范下的传播子如下:

$$\frac{i\mathcal{P}^{\alpha\beta,\gamma\delta}}{q^2 + i\varepsilon},\ \mathcal{P}_{\alpha\beta;\gamma\delta} = \frac{1}{2}\left[\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\beta\gamma}\eta_{\alpha\delta} - \eta_{\alpha\beta}\eta_{\gamma\delta}\right]. \tag{2}$$

Expressions for three-point-graviton vertices are derived in $[60-62, 64]$ and with background field theory vertices in [65-67]. Ghosts can be accounted for by the Faddeev-Popov approach.

三引力子顶点的表达式在 $[60-62, 64]$ 中给出，背景场论下的顶点可见文献 [65-67]。鬼场可以通过法捷耶夫-波波夫方法处理。

In this manner, one can regard perturbative computations in gravity as in traditional particle physics. Nevertheless, such a quantization steers into ultraviolet renormalization issues due to the dimensionful gravitational coupling G_N . It was demonstrated explicitly in 1974 by 't Hooft and Veltman [68], who directly evaluated the one-loop ultraviolet divergence and uncovered the following result:

通过这种方式，我们可以像传统粒子物理一样对引力做微扰计算。但由于引力耦合 G_N 具有量纲，该量子化方法会遇到紫外可重整性问题。't Hooft 和 Veltman 在 1974 年通过直接计算一圈紫外发散得到了明确结论，结果如下:

$$\mathcal{L}_{\text{UV divergence}} = \frac{1}{8\pi^2}\frac{1}{D-4}\left[\frac{1}{120}R^2 + \frac{7}{120}R^{\mu\upsilon}R_{\mu\upsilon}\right]. \tag{3}$$

Here, D represents the space-time dimension. A new divergence emerges at each order in perturbation theory, causing the theory to be non-renormalizable. In the context of effective field theory, the resolution assumes that the Hilbert-Einstein term in (1) is the leading term in a Lagrangian expansion that includes every conceivable generally covariant function of the metric and its derivatives. We develop the effective Lagrangian in the invariants ordered in the magnitude of their derivative contributions. It can be formally understood [69] as the expansion obtained from integrating out heavy mass degrees of freedom in favor of light in a fundamental theory below a specific scale. We thus introduce new higher-derivative couplings in terms of an infinite series of effective operators, which innately leads [70] to treat general relativity as an effective field theory.

此处 D 代表时空维度。微扰论的每一阶都会出现新的发散，因此该理论是不可重整的。在有效场论的框架下，解决思路是: 爱因斯坦-希尔伯特项只是拉格朗日展开的领头项，拉格朗日中包含所有由度规及其导数构成的广义协变函数。我们将有效拉格朗日按导数贡献的大小对不变量排序。该展开可以形式化理解 [69] 为: 在基本理论中，积分掉特定能标以下的大质量自由度，仅保留轻自由度后得到的展开。因此我们引入了由无穷多有效算符组成的高阶导数耦合项，这自然使得广义相对论被处理为有效场论 [70]。

$$\mathcal{L}_{\text{eff}} = \sqrt{-g}\left[\frac{R}{16\pi G_N} + C_1 R^2 + C_2 R^{\mu\upsilon}R_{\mu\upsilon} + \ldots\right]. \tag{4}$$

From a low-energy effective viewpoint, all the uncertainties about the ultraviolet completion of the theory are retained in the coefficients C_i , which have to be specified by experiments and observations. As a classical theory, it is essential to recognize that the above Lagrangian describes a theory of general relativity that is augmented by new operators, even as a classical theory. When considering effective actions, there is the question of the magnitudes of the coefficients in the derivative expansion in (4). Weinberg explained that any theory of massless spin-2 reduced to Einstein's gravity at large distances [71], and it has been asserted in [72,73] that the coefficients are parametrically suppressed by some higher-spin scalar M to be specified.

从低能有效理论的角度来看，理论关于紫外完备性的所有不确定性都被吸收进系数 C_i 中，这些系数需要通过实验和观测确定。需要明确的是，即使作为经典理论，上述拉格朗日描述的也是引入了新算符的广义相对论。讨论有效作用量时，会涉及 (4) 式导数展开中系数的大小问题。温伯格曾解释，任何无质量自旋 2 场的理论在长距离下都会约化为爱因斯坦引力 [71]，同时文献 [72,73] 指出，这些系数会被某个待定的更高能标标量 M 参数化压低。

The coefficient C_1 of the Ricci scalar squared term in (4) is dimensionless. From measurements of the absence of derivation from the $1/r$ law, the Eöt-Wash collaboration [74] inferred that C_1 could take very large value $0 \le C_1 < 10^{61}$. The conceivable presence of such a large coefficient in the effective action could be alarming, but this is not the case if it is regarded in the context of an $f(R)$ theory of gravity, where the C_1 decides that the mass of the scalaron $m^2 = 8\pi G_N/C_1$ [75]. For a value $C_1 \sim 10^{61}$, the scalaron has a mass of the order $10^{-3}eV/c^2$ like the neutrinos [76].

式 (4) 中里奇标量平方项的系数 C_1 是无量纲的。基于未观测到对 $1/r$ 定律的偏离，Eöt-Wash 合作组 [74] 推断 C_1 可以取非常大的值 $0 \le C_1 < 10^{61}$ 。有效作用量中存在如此大的系数看似值得警惕，但如果从引力的 $f(R)$ 理论语境来看就并非如此，在该理论中 C_1 决定了标量子 $m^2 = 8\pi G_N/C_1$ 的质量 [75]。当取值为 $C_1 \sim 10^{61}$ 时，标量子的质量量级为 $10^{-3}eV/c^2$ ，和中微子的质量量级相当 [76]。

We note that gravitational wave observations open an intriguing quest to further bound such coefficients by comparing observation to theory, particularly in the context of multi-messenger events [77,78]. Here, the effective field theory treatment must separate the long-range infrared and short-distance ultraviolet physics contributions permitting the extraction of long-range, low-energy results in the theory independent of the high-energy behavior of the theory.

我们注意到，引力波观测开启了一项有趣的研究: 通过将观测与理论对比进一步约束这些系数，在多信使天体物理事件的研究背景下尤其如此 [77,78]。在这里，有效场理论处理必须区分长程红外物理贡献和短程紫外物理贡献, 从而能够在不依赖理论高能行为的前提下提取出理论的长程低能结果。

Matter Couplings

物质耦合

Including matter in the effective treatment of general relativity can be accomplished by considering the Lagrangian

将物质纳入广义相对论的有效处理可通过考虑拉格朗日量实现

$$\mathcal{L} = \mathcal{L}_{\text{eff}} + \mathcal{L}_{\text{matter}}, \tag{5}$$

where the effective gravitational Lagrangian (4) is minimally coupled to the matter field $\sqrt{-g}\mathcal{L}_{\text{matter}} = g^{\mu\upsilon}T_{\mu\upsilon}$. It includes, in a covariant way, matter couplings. The rationale for following this route is that the gravitational interactions between two Schwarzschild black holes are obtained from gravitational scattering amplitudes between scalar fields minimally coupled to gravity by their stress-energy tensor

其中有效引力拉格朗日量 (4) 与物质场 $\sqrt{-g}\mathcal{L}_{\text{matter}} = g^{\mu\upsilon}T_{\mu\upsilon}$ 实现最小耦合。它以协变的方式包含了物质耦合。采用这一方案的依据是，两个史瓦西黑洞之间的引力相互作用，可通过引力散射振幅得到——该振幅来自通过自身能量动量张量与引力最小耦合的标量场之间的散射

$$T_{\mu\upsilon} \equiv \partial_\mu\phi\partial_\upsilon\phi - \frac{\eta_{\mu\upsilon}}{2}\left(\partial^\rho\phi\partial_\rho\phi - m^2\phi^2\right),\ \text{with}\ \sqrt{-g}\mathcal{L}_{\text{matter}} \equiv g^{\mu\upsilon}T_{\mu\upsilon}.$$

(6)

The multipole expansion is provided at the first post-Minkowskian order in [31] and the second post-Minkowskian order for aligned spins with the orbital angular momentum [34]. The multipole expansion from the spin of the Kerr black hole demands external states with spin. In the case of spin $S = 0, 1/2, 1$ [65, 79, 80] , spin $S = 2$ [81] computed the one-loop level. Since external states of spin, s , lead to a multipole expansion to the order $2s$, one must consider as well massive external states of higher spin [30 − 33, 36] , while for photons and massless fermions, we derive analogous stress-energy tensors. See, for instance, [82] for further details. Following the analysis accomplished using the techniques utilized in [65, 66, 70, 83] , where the following scalar vertices are uncovered in p_1 in and p_1' out-going convention. The two-scalar-one-graviton vertex $\tau_1^{\mu\upsilon}(p_1, p_1')$ is

多极展开已在 [31] 中给出一阶后闵可夫斯基近似结果，在 [34] 中给出轨道角动量对齐自旋的二阶后闵可夫斯基近似结果。克尔黑洞自旋带来的多极展开要求外态带有自旋。对于自旋 $S = 0, 1/2, 1$ [65, 79, 80]， 自旋 $S = 2$， 文献 [81] 计算了单圈水平。由于带自旋的外态 s 会导出 $2s$ 阶的多极展开，因此也必须考虑更高自旋的大质量外态 [30 − 33, 36]，而对于光子和无质量费米子，我们可以推导出类似的能量动量张量。更多细节参见例如 [82]。根据 [65, 66, 70, 83] 中所用技术完成的分析，在 p_1 入射和 p_1' 出射的 convention 下发现了如下标量顶点。两标量一引力子顶点 $\tau_1^{\mu\upsilon}(p_1, p_1')$ 为

$$\tau_1^{\mu\upsilon}(p_1, p_1') = \frac{i\kappa_{(4)}}{2}\left[p_1^\mu p_1'^\upsilon + p_1^\upsilon p_1'^\mu - \frac{1}{2}\eta^{\mu\upsilon}(p_1 + p_1')^2\right]. \tag{7}$$

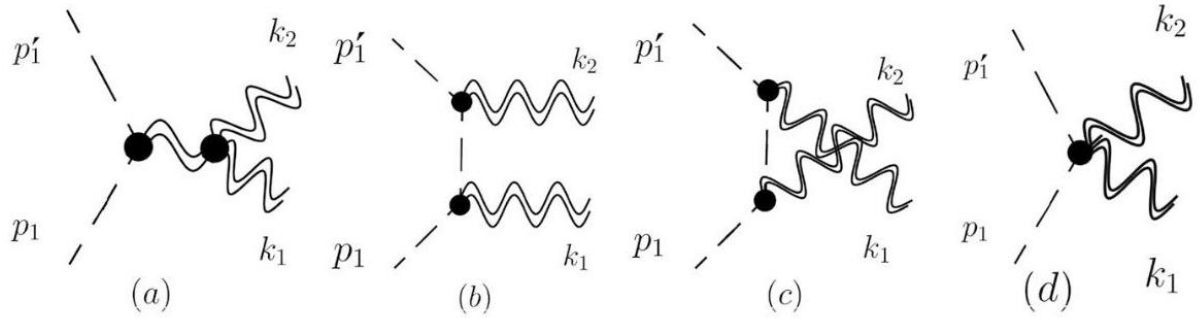


Fig. 3 The various contributions: (a) s -channel,(b) t -channel,(c) u -channel, and (d) contact term

图 3 各类贡献:(a) s 道，(b) t 道，(c) u 道，(d) 接触项

The two-scalar-two-graviton vertex $\tau_2^{\eta\lambda\rho\sigma}(p_1, p_1')$ is

两标量两引力子顶点 $\tau_2^{\eta\lambda\rho\sigma}(p_1, p_1')$ 为

$$\tau_2^{\eta\lambda\rho\sigma}(p_1, p_1') = -i\kappa_{(4)}^2 \left[\left\{ \mathcal{P}^{\eta\lambda,\alpha\delta}\mathcal{P}_\delta^{\rho\sigma,\beta} + \frac{1}{4}\left\{ \eta^{\eta\lambda}\mathcal{P}^{\rho\sigma,\alpha\beta} + \eta^{\rho\sigma}\mathcal{P}^{\eta\lambda,\alpha\beta} \right\} \right\} \right.$$

$$\left. \left(p_{1\alpha}p'_{1\beta} + p'_{1\alpha}p_{1\beta} \right) + \frac{1}{4}\mathcal{P}^{\eta\lambda,\rho\sigma}(p_1 + p_1')^2 \right]. \tag{8}$$

We evaluate all contributing diagrams at four points and compute the tree-level two-scalar-two-graviton amplitude (Fig. 3).

我们计算了所有四点贡献图，得到了树级两标量两引力子振幅 (图 3)。

Gravity Amplitudes from Quantum Field Theory

来自量子场论的引力振幅

As we have witnessed in the last section, we can regard perturbative computations of gravitational scattering amplitudes in the context of effective field theory. Early treatments of gravity relied on unwieldy off-shell vertex rules and led to somewhat unmanageable expressions due to the immense number of permutations and the many index contractions. This tremendous growth in complexity renders amplitude computations in pure gravity theory exceptionally problematic. Thus, it is advantageous that we have developed strategies in recent years where unitarity integrated with more painless on-shell remedies for tree amplitudes in gravity supplies pathways to streamline computations significantly.

正如我们在上一节所见，我们可以在有效场论框架下对引力散射振幅进行微扰计算。早期引力研究依赖繁琐的离壳顶点规则，由于排列组合数量极大且指标收缩众多，得到的表达式难以处理。复杂度的大幅增长使得纯引力理论中的振幅计算格外困难。因此，近年我们发展出的方法十分有利: 将幺正性与更简便的引力树振幅壳方法结合，为大幅简化计算提供了途径。

The benefit of using the spinor-helicity variable for external legs is more compact results for amplitudes. We will in this section outline such computations and the associated formalism. Generating amplitudes from Feynman graphs, one arrives at expressions with terms consisting of monomials of

对外腿使用旋量-螺旋度变量的好处是能得到更紧凑的振幅结果。本节我们将概述这类计算和相关形式体系。通过费曼图生成振幅，最终得到的表达式由单项式构成，单项式为

$$(k_i \cdot k_j),\ (\zeta_i \cdot \zeta_j),\ (\zeta_i \cdot k_j), \tag{9}$$

where k_i^μ is the momentum of the i th incoming leg and ζ_i^μ the associated polarization vector. In principle, the polarization vectors ζ_i^μ can be arbitrary, but in gauge theories, we have learned that it leads to incredibly compact expressions to utilize spinor-helicity variables. In this formalism, we compute all amplitude configurations with polarization vectors for positive and negative helicity gluons defined by

其中 k_i^μ 是第 i 个入射外腿的动量，ζ_i^μ 是对应的极化矢量。原则上，极化矢量 ζ_i^μ 可以任意，但我们在规范场论中已经了解到，使用旋量-螺旋度变量能得到极其紧凑的表达式。在该形式体系中，我们对所有振幅构型进行计算，其中正负螺旋度胶子的极化矢量定义为

(10)

$$\left(\zeta_i^+\right)_\mu\left(k_i, k_{\text{ref},i}\right) \equiv \frac{\left\langle k_{\text{ref},i}\left|\gamma_\mu\right| k_i\right\rangle}{\sqrt{2}\left\langle k_{\text{ref},i} k_i\right\rangle}$$

$$\left(\zeta_i^-\right)_\mu\left(k_i, k_{\text{ref},i}\right) \equiv \frac{-\left[k_{\text{ref},i}\left|\gamma_\mu\right| k_i\right\rangle}{\sqrt{2}\left[k_{\text{ref},i} k_i\right]}.$$

We have here employed

我们在此使用了

$$\left|k_j\right\rangle \equiv u_+\left(k_j\right) \equiv\left[\frac{1}{2}\left(1+\gamma_5\right)\right] u\left(k_j\right),$$

(11)

$$\left\langle k_j\right| \equiv \bar{u}_-\left(k_j\right) \equiv u\left(k_j\right)\left[\frac{1}{2}\left(1+\gamma_5\right)\right], \tag{12}$$

$$\left|k_j\right] \equiv u_-\left(k_j\right) \equiv\left[\frac{1}{2}\left(1-\gamma_5\right)\right] u\left(k_j\right), \tag{13}$$

$$\left[k_j\right| \equiv \bar{u}_-\left(k_j\right) \equiv \bar{u}\left(k_j\right)\left[\frac{1}{2}\left(1-\gamma_5\right)\right], \tag{14}$$

furnished with spinor solutions to Dirac’ s equation represented by $u\left(k_j\right)$ and $\bar{u}\left(k_j\right)$. We remark that the reference momentum in the overhead expressions is arbitrary. Specifying the spinor products

配有用 $u\left(k_j\right)$ 和 $\bar{u}\left(k_j\right)$ 表示的狄拉克方程旋量解。我们注意到，上述表达式中的参考动量是任意的。指定旋量积

$$\bar{u}_-\left(k_i\right) u_+\left(k_j\right)=\langle i j\rangle,\ \bar{u}_+\left(k_i\right) u_-\left(k_j\right)=[i j], \tag{15}$$

we have realized that in gauge theories, one can magically express n -point amplitudes in one simple formula dubbed as the Parke-Taylor amplitude.

我们已经发现，在规范场论中，人们可以用一个简单公式巧妙地表示 n 点振幅，该公式被称为帕克-泰勒振幅。

$$A_n^{\text{tree}}\left(1^-, 2^-, \ldots, n^+\right)=i \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots\langle n 1\rangle}. \tag{16}$$

We utilize this formalism in gravity by employing the concept of building gravity amplitudes using the KLT technique. It has its roots in the manner string theory amplitudes are organized perturbatively as pioneered by Bern et al. in numerous gravity contexts [84].

我们在引力中应用该形式体系，采用 KLT 技术构造引力振幅。该方法起源于弦论振幅的微扰组织方式，由 Bern 等人在众多引力场景中率先开展研究 [84]。

The KLT relations for gravity amplitudes take the following generic form for n -point scattering using a compact formulation in terms of the S -kernel [85, 86] . Here, $A^{\text{ tree }}(n, n-1, \sigma(2, \cdots, n-2), 1)$ signifies gauge theory amplitudes.

对于 n 点散射，引力振幅的 KLT 关系可以借助 S 核 [85, 86] 写成如下通用紧凑形式。此处 $A^{\text{ tree }}(n, n-1, \sigma(2, \cdots, n-2), 1)$ 代表规范理论振幅。

$$iM^{\text{ tree }}=\sum_{\tau, \gamma \in S_{n-3}} \mathcal{S}[\tau(2, \cdots, n-2) \mid \tau(2, \cdots, n-2)]|_{k_1}$$

$$\times A_n^{\text{tree }}(1, \tau(2, \cdots, n-2), n-1, n) A_n^{\text{tree }}(n, n-1, \sigma(2, \cdots, n-2), 1),$$

(17)

with S_{n-3} representing the possible permutations over $n-3$ indices and where the momentum kernel $\mathcal{S}$ is supplied by the expression ($\theta\left(i_t, i_s\right)$ equals 1 if the ordering of the legs i_r and i_s is opposite in the sets $\{i_1, \ldots, i_r\}$ and $\{j_1, \ldots, j_r\}$ and 0 if the ordering is the same.

其中 S_{n-3} 是 $n-3$ 个指标上所有可能的排列，动量核 $\mathcal{S}$ 满足: 当外腿 i_r 和 i_s 在集合 $\{i_1, \ldots, i_r\}$ 和 $\{j_1, \ldots, j_r\}$ 中的排序相反时，($\theta\left(i_t, i_s\right)$ 等于 1；排序相同时等于 0。

$$\mathcal{S}[i_1, \ldots, i_r \mid j_1, \ldots, j_r]|_p=\prod_{t=1}^r\left(p \cdot k_{i_r}+\sum_{s>t}^r \theta\left(i_r, i_s\right) k_{i_r} \cdot k_{i_s}\right). \tag{18}$$

Thus, utilizing this formula, n -point gravity amplitudes $M_n^{\text{tree }}$ can be delivered from color-ordered amplitudes in gauge theory $A_n^{\text{tree }}(\sigma)$. We favor the above form of the KLT relations in unitarity computations, but it is critical to recognize that multiple identical rewritings of the KLT relations are feasible. It is due to the equivalences that follow from

因此，利用该公式，n 点引力振幅 $M_n^{\text{tree }}$ 可以从规范理论中的色序振幅 $A_n^{\text{tree }}(\sigma)$ 得到。我们在幺正性计算中偏好 KLT 关系的上述形式，但必须注意，KLT 关系存在多种等价改写形式，这一点至关重要。这些等价改写来自于以下导出的等价性:

$$\sum_{n \in S_{n-2}} \mathcal{S}[\tau(2, \cdots, n-1) \mid \sigma(2, \cdots, n-2)]|_{k_1}$$

$$\times A_n^{\text{tree }}(1, \tau(2, \cdots, n-1), n)=0$$

$$\forall \sigma \in S_{n-2} \tag{19}$$

Thus, with the KLT approach, we can efficiently evaluate the emission of two gluons from massive scalars without considering graviton vertex rules. From these relations, it is possible to derive helicity expressions for graviton amplitudes and employ them as input in generalized unitarity cuts.

因此，借助 KLT 方法，我们可以无需考虑引力子顶点规则，高效计算大质量标量的双胶子辐射过程。从这些关系出发，可以推导出引力子振幅的螺旋度表达式，并将其作为广义幺正切割的输入。

We generate very compact identities for graviton amplitudes by exploiting the spinor-helicity formalism. Graviton helicity polarizations can be inferred from gluon polarizations as follows:

通过利用旋量-螺旋度形式体系，我们得到了引力子振幅的非常紧凑的恒等式。引力子螺旋度极化可以从胶子极化按下式推导得到:

$$\left(\xi_i^+\right)_{\mu\upsilon}\left(k_i, k_{\mathrm{ref},i}^L, k_{\mathrm{ref},i}^R\right) = \left(\zeta_i^+\right)_\mu\left(k_i, k_{\mathrm{ref},i}^L\right) \times \left(\zeta_i^+\right)_\upsilon\left(k_i, k_{\mathrm{ref},i}^R\right) \tag{20}$$

$$\left(\xi_i^-\right)_{\mu\upsilon}\left(k_i, k_{\mathrm{ref},i}^L, k_{\mathrm{ref},i}^R\right) = \left(\zeta_i^-\right)_\mu\left(k_i, k_{\mathrm{ref},i}^L\right) \times \left(\zeta_i^-\right)_\upsilon\left(k_i, k_{\mathrm{ref},i}^R\right).$$

Employing these polarization vectors for computations of the two-scalar-two-graviton amplitude, we attain the compact expressions

将这些极化矢量用于两标量-两引力子振幅的计算，我们得到了紧凑的表达式

$$iM_0^{\mathrm{tree}}\left(p_1, p_1', k_1^+, k_2^+\right) = \frac{\kappa_{(4)}^2}{16}\frac{1}{(k_1\cdot k_2)}\frac{m^4[k_1k_2]^4}{(k_1\cdot p_1)(k_1\cdot p_1')},$$

$$iM_0^{\mathrm{tree}}\left(p_1, p_1', k_1^-, k_2^+\right) = \frac{\kappa_{(4)}^2}{16}\frac{1}{(k_1\cdot k_2)}\frac{\langle k_1|p_1|k_2\rangle^2\langle k_1|p_1'|k_2\rangle^2}{(k_1\cdot p_1)(k_1\cdot p_1')}, \tag{21}$$

with $iM_0^{\mathrm{tree}}\left(p_1, p_2, k_1^-, k_2^-\right) = \left(iM_0^{\mathrm{tree}}\left(p_1, p_2, k_1^+, k_2^+\right)\right)^*$ and $iM_0^{\mathrm{tree}}\left(p_1, p_2, k_1^+, k_2^-\right) = \left(iM_0^{\mathrm{tree}}\left(p_1, p_2, k_1^-, k_2^+\right)\right)^*$.

其中 $iM_0^{\mathrm{tree}}\left(p_1, p_2, k_1^-, k_2^-\right) = \left(iM_0^{\mathrm{tree}}\left(p_1, p_2, k_1^+, k_2^+\right)\right)^*$ 和 $iM_0^{\mathrm{tree}}\left(p_1, p_2, k_1^+, k_2^-\right) = \left(iM_0^{\mathrm{tree}}\left(p_1, p_2, k_1^-, k_2^+\right)\right)^*$.

For the two-scalar-three-graviton amplitude $(+++)$ configuration, we see that

对于双标量三引力子振幅 $(+++)$ 构型，我们可以得到

$$i\mathcal{M}_0^{\mathrm{tree}}\left(p_1, p_1', k_1^+, k_2^+, k_3^+\right) =$$

$$\left(-\frac{(8\pi G_N)^{\frac{3}{2}}m_1^4}{\langle k_1k_2\rangle^2\langle k_1k_3\rangle^2\langle k_2k_3\rangle^2}\sum_{1\leq i\neq j\neq k\leq 3}\frac{(k_i\cdot k_j)(k_j\cdot k_k)\,\mathrm{tr}_+\left[k_k, p_1, p_1', k_i\right]}{(p_1\cdot k_k)(p_1'\cdot k_i)}\right), \tag{22}$$

with $i\mathcal{M}_0^{\mathrm{tree}}\left(p_1, p_1', k_1^-, k_2^-, k_3^-\right)$ obtained by complex conjugation. This expression agrees with [87] and vanishes as expected when $m_1 = 0$. We have defined

其中 $i\mathcal{M}_0^{\text{tree}}\left(p_1, p_1', k_1^-, k_2^-, k_3^-\right)$ 由复共轭得到。该表达式与文献 [87] 一致，且当 $m_1 = 0$ 时符合预期等于零。我们定义了

$$tr_\pm\left(abcd\right) \equiv 2\left(a\cdot bc\cdot d - a\cdot cb\cdot d + a\cdot db\cdot c\right) \pm 2i\varepsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma. \tag{23}$$

For the $(-++)$ amplitude, we get

对于 $(-++)$ 振幅，我们得到

$$\begin{aligned}
&i\mathcal{M}_0^{\text{tree}}\left(p_1, p_1', k_1^-, k_2^+, k_3^+\right)\\
&= \frac{(2\pi G_N)^{\frac{3}{2}}}{2}\left(\sum_{2\le j\ne k\le 3}\frac{\langle k_1|p_1\ k_j]\langle k_1|p_1'\ k_j]^2\langle k_1|p_1\mid k_k]^3}{\langle k_1k_j\rangle\langle k_1k_k\rangle(k_1\cdot k_j)(k_1\cdot k_k)(p_1\cdot k_1)(p_1'\cdot k_j)}\right.\\
&\quad-\frac{\langle k_1|p_1|k_2]^3\langle k_1|p_1'\rangle k_3]^3}{\langle k_1k_2\rangle\langle k_1k_3\rangle(k_1\cdot k_2)(k_1\cdot k_3)(p_1\cdot k_2)(p_1'\cdot k_3)}\\
&\quad-\frac{2\,[k_2k_3]\langle k_1|p_1|k_2\rangle\langle k_1|p_1|k_3\rangle\langle k_1|p_1|\ p_1'\mid k_1\rangle^2}{\langle k_1k_2\rangle\langle k_1k_3\rangle\langle k_2k_3\rangle(k_1\cdot k_2)(k_1\cdot k_3)(p_1\cdot k_1)}\\
&\quad\left.+\frac{2[k_2k_3]^3\langle k_1|p_1|\ p_1'\mid k_1\rangle^2}{\langle k_2k_3\rangle(k_1\cdot k_2)(k_1\cdot k_3)\,t}\right)+(p_1\leftrightarrow -p_1'),
\end{aligned}\tag{24}$$

with $i\mathcal{M}_0\left(p_1, p_1', k_1^+, k_2^-, k_3^-\right)$ obtained by complex conjugation.

其中 $i\mathcal{M}_0\left(p_1, p_1', k_1^+, k_2^-, k_3^-\right)$ 由复共轭得到。

We will assess how these tree amplitudes are helpful when we employ unitarity cuts as a route to loop amplitudes. We comment in passing that the above amplitude examples would be highly cumbersome to derive straight from the vertex rules. It demonstrates the power of the double-copy KLT techniques fused with spinor-helicity. An exciting perspective arises when considering the Kawai-Lewellen-Tye

我们将分析，当我们利用幺正切割方法构造圈振幅时，这些树图振幅有何帮助。顺便提一句，若直接从顶点规则出发推导上述振幅实例，会极为繁琐。这体现了结合旋量螺旋度的双拷贝 KLT 技术的强大威力。当研究川合-勒韦伦-泰 (

(KLT) relations in the context of effective field theory extensions. In many cases, it is possible to maintain KLT factorizations of higher-derivative terms in effective generic Lagrangians for gauge theories and gravity. It opens up more straightforward amplitude computations of extensions of Einstein's gravity. One finds details on such calculations in Refs. [88-91].

KLT) 关系在有效场论扩展框架下的应用时，我们得到了一个令人振奋的研究方向: 在很多情况下，对于规范理论和引力的一般有效拉氏量，高阶导数项仍然可以保持 KLT 因子化性质，这为爱因斯坦引力扩展理论的振幅计算提供了更简便的途径。读者可以在文献 [88-91] 中找到这类计算的细节。

New On-Shell Methods and Double-Copy and BCJ Numerators

新的在壳方法、双拷贝与 BCJ 分子

Current techniques have simplified the framework for the computation of tree amplitudes of gravitational interactions. We will, in this section, discuss how we can enable an even more compact construction for graviton amplitudes from color-kinematic numerators. In particular, we will establish how we can derive classical results in pure gravity in a dimension-agnostic way. A path to this framework is from the computation of tree-level multi-graviton emission from a scalar line using the scattering equations procedure pioneered by Refs. [92-95]. This formalism reaches scattering amplitudes automatically satisfying color-kinematic dualities for a large class of field theories in a remarkably compact manner. We can develop amplitudes for gluon scattering acquired from contour integrals. See [96] for additional details. The important ingredient in such formulations is the production of numerators $N_{n-2}\left(1,\beta\left(2,\ldots,n-1\right),n\right)$ which we can write in terms of tensor products of gluon polarization vectors and momenta. One can develop several efficient schemes for computing such numerators; see, e.g., [97-100]. We thus arrive at the scattering amplitude

现有技术已经简化了引力相互作用树图振幅的计算框架。在本节中，我们将讨论如何从色运动学分子出发，构造更为紧凑的引力子振幅。我们尤其会阐明如何在不依赖维度的前提下得到纯引力中的经典结果。该框架可以追溯到利用由文献 [92-95] 开创的散射方程方法，计算标量线辐射出多个引力子的树级过程。该形式化方法能自动得到满足一大类场论色运动学对偶性的散射振幅，形式极为简洁。我们可以得到从围道积分获得的胶子散射振幅，更多细节见文献 [96]。这类构造的核心是得到分子 $N_{n-2}\left(1,\beta\left(2,\ldots,n-1\right),n\right)$，我们可以将其写为胶子极化矢量与动量张量积的形式。目前已经发展出多种计算这类分子的高效方案，例如见文献 [97-100]。由此我们得到散射振幅

$$A_{n-2}^{\text{tree}}\left(1,\sigma\left(2,\ldots,n-1\right),n\right)=\sum_{\gamma\in S_{n-2}}\mathcal{S}^{-1}\left(\beta\gamma\right)\big|_{p_1}N_{n-2}\left(1,\sigma\left(2,\cdots,n-1\right),n\right),$$

(25)

where $S^{-1}\left(\beta\gamma\right)\big|_{p_1}$ denotes the associated denominator poles. This function is the inverse momentum kernel. Using this in the context of gravity using the KLT relations, one arrives at

其中 $S^{-1}\left(\beta\gamma\right)\big|_{p_1}$ 表示关联的分母极点，该函数是逆动量核。在引力中结合 KLT 关系使用该方法，可以得到

$$M_{n-2}^{\text{tree}}\left(1,2,\ldots,n\right)=i\sum_{\sigma\in S_{n-2}}N_{n-2}\left(1,\sigma\left(2,\cdots,n-1\right),n\right)A_{n-2}\left(1,\sigma\left(2,\ldots,n-1\right),n\right).$$

(26)

A compact starting point for the computation of tree graviton amplitudes. Poles, in the expression, arise only from the Yang-Mills amplitude on the right. From dimensional reduction, one can derive multi-gluon emission from a massive scalar from pure multi-gluon amplitudes. Thus, this formalism is remarkably malleable, and one can use it in the context of several different theories, for instance, with massive spin-1 or fermion states. See [100] for some additional details. Some numerator and amplitude examples are

这是计算树级引力子振幅的紧凑出发点。表达式中的极点仅来自右侧的杨-米尔斯振幅。通过维约化，我们可以从纯多胶子振幅得到有质量标量辐射多胶子的过程。因此该形式化方法灵活性极强，可应用于多种不同理论，例如包含有质量自旋 1 粒子或费米子的理论，更多细节见文献 [100]。部分分子与振幅示例如下

$$M_1^{\text{tree}}\left(p, \ell_2, -p'\right) = iN_1\left(p, \ell_2, -p'\right) A_1\left(p, \ell_2, -p'\right) = iN_1\left(p, \ell_2, -p'\right)^2,$$

(27)as well as

以及

$$M_2^{\text{tree}}\left(p, \ell_2, \ell_3, -p'\right) = iN_2\left(p, 2, 3, -p'\right) A_2\left(p, 2, 3, -p'\right) + \text{ perm. } \{2, 3\}$$

$$= \frac{iN_2\left(p, 2, 3, -p'\right)^2}{\left(\ell_2 + p\right)^2 - m^2 + i\varepsilon} + \frac{iN_2\left(p, 3, 2, -p'\right)^2}{\left(\ell_3 + p\right)^2 - m^2 + i\varepsilon} + \frac{i\left(N_2^{[2,3]}\right)^2}{\left(\ell_2 + \ell_3\right)^2 + i\varepsilon}.$$

(28)

We have comparable expressions at higher multiplicities. We direct to our online repository for expressions for numerator factors. We emphasize that amplitudes derived this way are accurate in arbitrary dimensions, satisfy manifestly color-kinematic relations, and incorporate no spurious poles. We can equally well generate expressions for multi-photon emission from a massive charged scalar exploiting the photon decoupling identity.

更高多重度下也存在类似表达式，分子因子的表达式可查阅我们的在线代码仓库。需要强调的是，通过该方法得到的振幅在任意维度下都精确成立，显式满足色运动学关系，且不包含伪极点。利用光子退耦恒等式，我们还可以同样得到有质量带电标量辐射多光子的表达式。

$$A_{n-2}^{\text{photon}}\left(p, 2, \ldots, n-1, -p'\right) = \sum_{\beta \in S_{n-2}} A_{n-2}^{\text{tree}}\left(p, \beta\left(2, \ldots, n-1\right), -p'\right). \tag{29}$$

We utilize the above formula for computations in scalar QED.

我们将上述公式用于标量量子电动力学的计算中。

Remodeling Gravitational Amplitudes

重塑引力振幅

The emphasis here is on applying modern amplitude methods for evaluating the gravitational interaction between two bodies and their association with gravitational physics.

本文此处重点在于应用现代振幅方法计算两体间的引力相互作用，以及它们与引力物理的关联。

Current advancement has materialized from utilizing generalized unitarity. In such studies, we require to acquire (1) an ansatz for the considered amplitude in the context of a basis of master integrals and (2) coefficients for each master integral assessed by considering unitarity conditions. The earlier section witnessed how to compute input trees for unitarity efficiently. The recent improvement in computing on-shell amplitudes permits evaluating loop amplitudes at high perturbative orders, thereby acquiring relativistic invariant results valid in all energy regimes from small relative velocities to the ultra-relativistic limit [9, 17, 22] . Deriving the classical post-Minkowskian results as a component of the total quantum gravity S -matrix between two massive bodies opens a new stance on the subtle queries, for instance, on gravitational radiation and ultra-high-energy scattering in classical gravity. The scattering amplitude method completes the post-Newtonian computations by delivering information exceeding its regime of validity and leads to astonishing results joining the conservative part and gravitational radiation effects [9, 16, 18 − 22, 101] . It offers a new outlook on the traditional methods [57, 102-108] employed to compute the gravitational wave templates. The same formalism unites the re-summed post-Newtonian results [13, 15, 26, 27] and the high-energy behavior [17, 20] .

目前的进展来自广义幺正性的应用。这类研究中，我们需要得到:(1) 主积分基下待研究振幅的假设形式；(2) 由幺正性条件确定的每个主积分的系数。前文已经介绍了如何高效计算幺正性所需的树图输入。近年来对壳振幅计算的改进使得我们可以计算高微扰阶的圈振幅，进而得到适用于从小相对速度到极端相对论极限所有能区的相对论不变结果 [9, 17, 22] 。从全量子引力 S 矩阵的两体散射结果中提取经典后闵可夫斯基结果，为研究引力辐射、经典引力超高能散射这类微妙问题提供了全新视角。散射振幅方法通过给出超出后牛顿近似有效范围的信息补充了后牛顿计算，并且得到了将保守部分与引力辐射效应结合的惊人结果 [9, 16, 18 − 22, 101] 。它为计算引力波模板的传统方法 [57, 102-108] 提供了全新视角。这套形式体系统一了重求和后的后牛顿结果 [13, 15, 26, 27] 和高能行为 [17, 20] 。

Another attractive characteristic is that the scattering amplitude approach applies to any space-time dimension. It is especially intriguing since black holes in higher dimensions offer fascinating new features [109].

散射振幅方法另一个吸引人的特点是，它适用于任意时空维度。这一点尤其有趣，因为高维黑洞拥有引人入胜的新特性 [109]。

The extraction of the classical general relativity contribution from scattering amplitudes has guided a reformulation of the classical $\hbar \to 0$ limit [22] of scattering amplitudes that differ from the logic of the naïve textbook $\hbar$ counting, e.g.,[110]. This approach is an excellent exposition of the statement by Kovacs and Thorne in [111] that ”Any classical problem can be solved quantum-mechanically; and sometimes, the quantum solution is easier than the classical.”

从散射振幅中提取经典广义相对论贡献的工作，引导我们重新表述散射振幅的经典 $\hbar \to 0$ 极限，这套表述不同于传统教科书中朴素的 $\hbar$ 计数逻辑，例如见 [110]。这套方法很好地阐释了科瓦奇和索恩在 [111] 中的论断:「任何经典问题都可以通过量子力学求解，有时量子解法比经典解法更简单。」

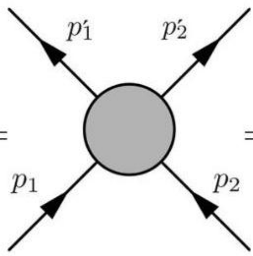
Setting Up a Convenient Formalism

建立方便的形式体系

We will now consider establishing a convenient convention for computations. We are interested in extracting physical observables from the gravitational interactions between two massive body of masses m_i and spin S_i with $i = 1, 2$ interacting through the exchange of massless spin-2 gravitons [60-63,68] (One could as well as include electromagnetic interactions as considered in [83, 112]). The vital limitation of the present analysis is that we have large external masses or charges with a massless exchange. We can regard observables in scalar QED as a testing ground for the gravitational interactions [113]. We can expand the two-body scattering matrix in a perturbation series

我们现在将考虑为计算建立一套方便的约定。我们的研究目标是从质量为 m_i 、自旋为 S_i 的两个大质量天体经 $i = 1, 2$ 通过交换无质量自旋 2 引力子发生的引力相互作用中提取物理可观测量 [60-63,68](也可以像文献 [83, 112] 中那样额外纳入电磁相互作用)。本分析的核心限制条件是我们研究的是大质量外质量或电荷、且交换的粒子为无质量的情况。我们可以将标量量子电动力学中的可观测量作为引力相互作用的检验基础 [113]。我们可以将两体散射矩阵按微扰级数展开

$$\mathscr{M}(p_1, p_2, p_1', p_2') =$$



$$= \sum_{L=0}^{\infty} G_N^{L+1} \mathscr{M}_L(\gamma, q^2).$$

(30)

The quantum scattering matrix $\mathcal{M}(p_1, p_2, p_1', p_2')$ depends on the relativistic factor $\gamma := p_1 \cdot p_2/(m_1 m_2)$, the momentum transfer $(p_1 - p_1')^2 =: q^2$, and $\hbar$.

量子散射矩阵 $\mathcal{M}(p_1, p_2, p_1', p_2')$ 依赖于相对论因子 $\gamma := p_1 \cdot p_2/(m_1 m_2)$、动量转移 $(p_1 - p_1')^2 =: q^2$ 以及 $\hbar$ 。

At a given order in perturbation, one obtains the exchange of gravitons (curly lines) between massive external matters (solid lines)

在微扰论的给定阶数下，可得到引力子 (波浪线) 在大质量外物质 (实线) 之间的交换过程

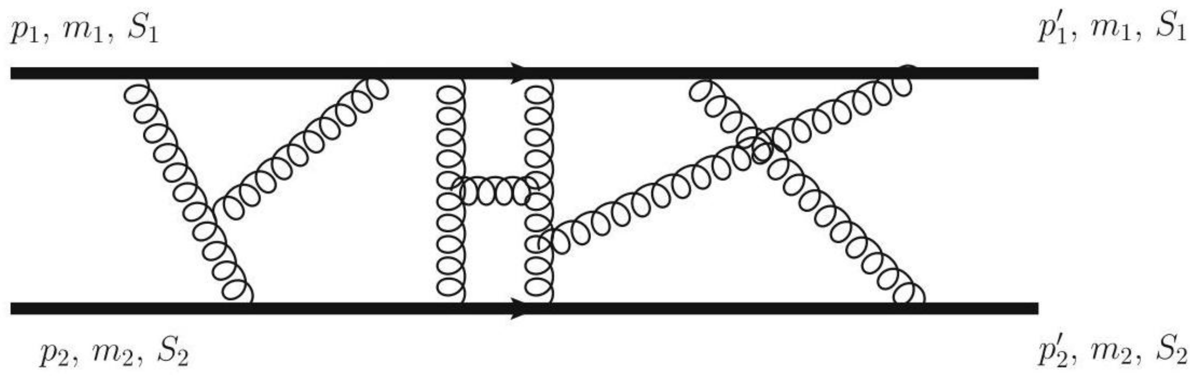
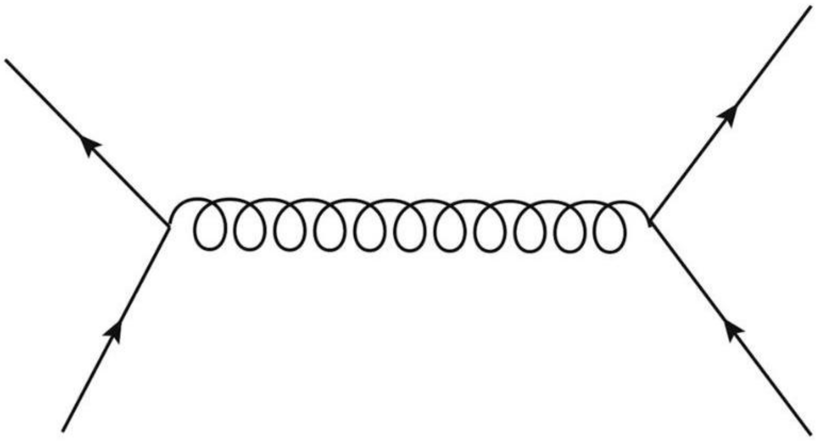


Fig. 4 Gravitational two-body scattering at tree level

图 4 树阶引力两体散射



The Examples of Tree-Level and One-Loop Amplitudes

树图阶和单圈振幅示例

We illustrate the emergence of the classical and quantum pieces at tree-level and one-loop order. The tree-level scattering between two massive fields in Fig. 4 has the $\hbar$ expansion

我们阐述经典项和量子项如何出现在树图阶和单圈阶。图 4 中两个大质量场的树图散射具有 $\hbar$ 展开式

$$\mathcal{M}_0\left(\gamma, \underline{q}^2, \hbar\right) = \frac{-16\pi G_N m_1^2 m_2^2 \left(2\gamma^2 - 1\right)}{\hbar \left|\underline{q}\right|^2} + \hbar 4\pi G_N p_1 \cdot p_2. \tag{31}$$

In this expression, we recognize the first post-Minkowskian classical contribution $G_N m_1^2 m_2^2 \left(2\gamma^2 - 1\right)$ matching the results from general relativity (see, e.g.,[101]) and the higher-order quantum correction $\hbar p_1 \cdot p_2$ due to the contact term as mandated by the full quantum amplitude.

在该表达式中，我们可以识别出第一个后闵可夫斯基经典贡献 $G_N m_1^2 m_2^2 \left(2\gamma^2 - 1\right)$ 与广义相对论的结果匹配 (例如参见文献 [101])，而高阶量子修正 $\hbar p_1 \cdot p_2$ 来自全量子振幅要求的接触项。

In reference [7], we elucidated that generalized unitarity is an excellent tool to calculate terms that resemble long-range contributions in amplitudes. Such nonanalytic terms provide us with classical scattering potentials in theories such as QED, gravity, and quantum modifications. Since we are exclusively interested in non-polynomial contributions, we are not required to generate the full amplitude. Identifying those terms in the amplitude is adequate for classical and leading quantum corrections. Thus, a pathway is established to streamline such computations. At one-loop order, we fetch coefficients corresponding to $1/\sqrt{-q^2}$ and $\log\left(-q^2\right)$ terms in the amplitude from on-shell unitarity. Following the approach provisioned in [114], this can, e.g., be done through evaluating the phase-space integrals by reinstating the off-shell cut propagators with on-shell cut conditions in numerators.

在文献 [7] 中，我们阐明广义幺正性是计算振幅中类似长程贡献项的优秀工具。这类非解析项可以为 QED、引力等理论中的经典散射势以及量子修正提供结果。由于我们仅对非多项式贡献感兴趣，因此无需生成完整振幅。在振幅中识别出这些项就足以得到经典修正和领头阶量子修正。由此我们找到了简化这类计算的途径。在单圈阶，我们通过在壳幺正性提取振幅中对应 $1/\sqrt{-q^2}$ 和 $\log\left(-q^2\right)$ 项的系数。遵循文献 [114] 给出的方法，例如可以通过恢复满足在壳切割条件的分子 off-shell 切割传播子，对相空间积分完成计算。

Formally at one loop, we thus have to consider the cut associated with the integral expression (Fig. 5)

因此，在单圈形式下我们需要考虑积分表达式对应的切割 (图 5)

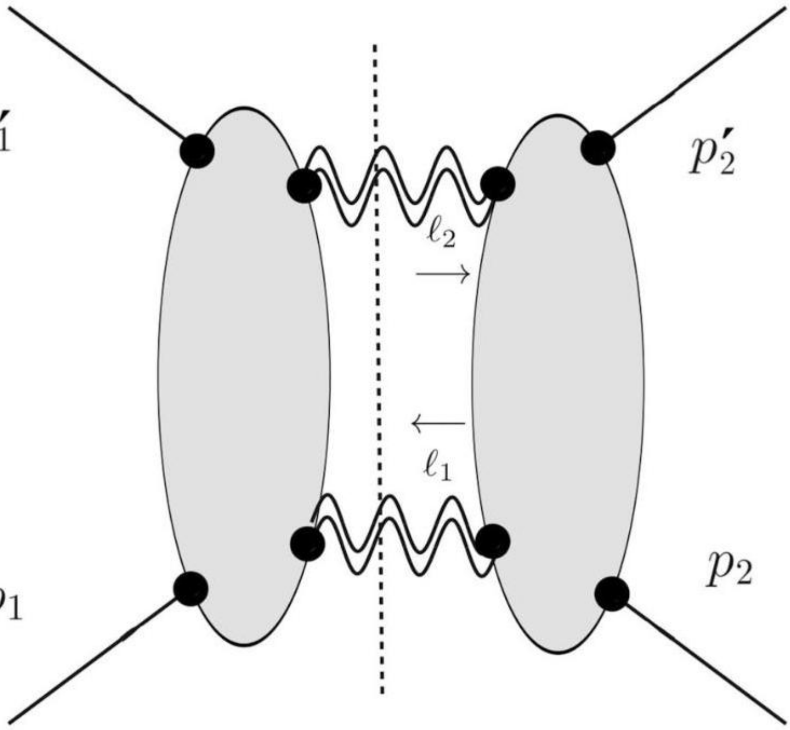
$$iM^{1-\text{ loop}}\big|_{\text{disc}}$$

$$= \int \frac{d^D\ell}{(2\pi)^D} \frac{\sum_{\lambda_1,\lambda_2} M^{\text{tree}}_{\lambda_1\lambda_2}\left(p_1, p_1', -\ell_2^{\lambda_2}, \ell_1^{\lambda_1}\right)\left(M^{\text{tree}}_{\lambda_1\lambda_2}\left(p_2, p_2', \ell_2^{\lambda_2}, -\ell_1^{\lambda_1}\right)\right)^*}{\ell_1^2\ell_2^2}\Bigg|_{\text{cut}} .$$

(32)

Fig. 5 The double cut

图 5 双切割



Here, we have the cut conditions $\ell_1^2 = \ell_2^2 = 0$, and we sum over all feasible physical graviton helicity arrangements across the cut: λ_1 and λ_2 . Box, triangle, and bubble graph topologies supply the basis for the quantum and classical contributions at one loop. In the cut, we can directly pinpoint the integral functions and thus isolate the coefficients for the non-analytic terms we are pursuing. In [23], we separated the classical non-analytically contributions by evaluating the triple cut and recognizing the coefficients for the two-loop basis of integral functions. We expand, at one-loop order, the full quantum two-body scattering amplitude on the standard basis of scalar one-loop integrals in four dimensions

此处我们得到切割条件 $\ell_1^2 = \ell_2^2 = 0$ ，并对切割上所有可行的物理引力子螺旋度组态求和: λ_1 和 λ_2 。箱图、三角图和泡图拓扑构成了单圈量子贡献和经典贡献的基。在切割中我们可以直接定位积分函数，从而分离出我们寻找的非解析项的系数。在文献 [23] 中，我们通过计算三切割分离了经典非解析贡献，并识别出两圈积分函数基的系数。我们在四维空间中对标量单圈积分的标准基展开了完整的量子两体散射振幅

$$\mathcal{M}_1 = \frac{i16\pi^2 G_N^2 m_1^2 m_2^2}{E_1 E_2}$$

$$\left(3\left(1-5\gamma^2\right)\left(m_1^2 I_{\triangleright} + m_2^2 I_{\triangleleft}\right) + 4m_1^2 m_2^2 \left(1-2\gamma^2\right)^2 \left(I_{\square} + I_{\bowtie}\right) + \cdots\right),$$

(33)

where $E_i^2 = \mathbf{p}_i^2 + m_i^2$ with $i = 1,2$ is the energy of the particle $i = 1,2$. The expression contains the massive scalar triangles, which have the large mass expansion exhibiting the classical $1/\hbar$ term [5, 10, 115]

其中 $E_i^2 = \mathbf{p}_i^2 + m_i^2$ 与 $i = 1,2$ 中，$i = 1,2$ 是粒子的能量。该表达式包含大质量标量三角图，其大质量展开给出经典 $1/\hbar$ 项 [5, 10, 115]

$$I_{\triangleright} = -\frac{i}{32m_1}\frac{1}{\hbar\left|\underline{q}\right|} + \cdots, \; I_{\triangleleft} = -\frac{i}{32m_2}\frac{1}{\hbar\left|\underline{q}\right|} + \cdots \tag{34}$$

and the scalar box and cross-box integrals

以及标量箱图和交叉箱图积分

$$I_{\square} = \frac{i}{16\pi^2\hbar^2\left|\underline{q}\right|^2}\left(-\frac{1}{m_1m_2} + \frac{m_1(m_1-m_2)}{3m_1^2m_2^2} + \frac{i\pi}{|p|(E_1+E_2)}\right)$$

$$\left(\frac{2}{3-d} - \hbar^2\log|q|^2\right) + \cdots$$

$$I_{\bowtie} = \frac{i}{16\pi^2\hbar^2\left|\underline{q}\right|^2}\left(\frac{1}{m_1m_2} - \frac{m_1(m_1-m_2)}{3m_1^2m_2^2}\right)\left(\frac{2}{3-d} - \hbar^2\log|q|^2\right) + \cdots$$

(35)

Putting everything together, we get that expansion of the total quantum one-loop amplitude read

将所有结果整合后，我们得到完整量子单圈振幅的展开式为

$$\mathcal{M}_1\left(\gamma, \underline{q}^2, \hbar\right) = \frac{\pi^2G_N^2m_1^2m_2^2}{E_1E_2}\Bigg[-\frac{3\left(1-5\gamma^2\right)}{2\hbar\left|\underline{q}\right|}(m_1+m_2)$$

$$+\frac{im_1m_2}{(E_1+E_2)}\frac{4\left(1-2\gamma^2\right)^2}{|\mathbf{p}|}\frac{\left(\frac{2}{3-d} - \hbar^2\log|q|^2\right)}{\pi\hbar^2\left|\underline{q}\right|^2}\Bigg] + \cdots. \tag{36}$$

This expression contains:

该表达式包含:

- At order $1/\hbar^2$, a contribution given by the square of the classical tree-level contribution from (31). This piece we need for the exponentiation of the S -matrix in (54) as detailed in [24].

  - 在 $1/\hbar^2$ 阶，贡献由式 (31) 中经典树图贡献的平方给出。如文献 [24] 所述，这一项是我们对式 (54) 中 S 矩阵做指数化所需要的。

- At order $1/\hbar$, the classical second post-Minkowskian contribution [10] that matches the classical second post-Minkowskian result for generic masses.

  - 在 $1/\hbar$ 阶，给出经典二阶后闵可夫斯基贡献 [10]，与任意质量下的经典二阶后闵可夫斯基结果匹配。

- A quantum piece of order $\hbar^0$ which is a long-range infrared quantum gravity effect. Because this is the first quantum correction to the classical result, the value of the quantum gravity-induced correction is universal and independent of the ultraviolet regularization [7, 82] .

- 一个 $\hbar^0$ 阶的量子项，是长程红外量子引力效应。由于这是对经典结果的第一阶量子修正，量子引力诱导修正的取值是普适的，与紫外正则化 [7, 82] 无关。

We will now generalize this discussion to any perturbative order. A traditional argument (see, for instance,[110]) gives that the L -loop contribution is of order $\mathcal{M}_L\left(\gamma, q^2\right) = \mathcal{O}\left(\hbar^{L-1}\right)$. A different behavior emerges when maintaining fixed wavenumber $q = q/\hbar$ and taking both the $\hbar \to 0$ and the small momentum transfer $q \to 0$ limit [10, 115, 116] . The L -loop two-body scattering amplitude has the Laurent expansion in four dimensions

我们现在将把该讨论推广到任意微扰阶。传统观点认为 (例如参见文献 [110])，L 圈贡献的阶为 $\mathcal{M}_L\left(\gamma, q^2\right) = \mathcal{O}\left(\hbar^{L-1}\right)$ 。当保持波数 $q = q/\hbar$ 固定，同时取 $\hbar \to 0$ 与小动量转移 $q \to 0$ 极限 [10, 115, 116] 时，会出现不同的行为。四维下 L 圈双体散射振幅的洛朗展开为

$$\mathcal{M}_L\left(\gamma, \underline{q}, \hbar\right) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, D)}{\hbar^{L+1}\left|\underline{q}\right|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, D)}{\hbar\left|\underline{q}\right|^{\frac{L(4-D)}{2}+2-L}} + O\left(\hbar^0\right). \tag{37}$$

Again, we can systematize the full quantum amplitude in terms of three types of contributions:

同样，我们可以将全量子振幅按三类贡献系统化整理:

1. The terms of order $1/\hbar^r$ with $L+2 \leq r \leq 2$ that are more singular than the classical piece in the $\hbar \to 0$ limit.

1. $\hbar \to 0$ 极限下，阶为 $1/\hbar^r$ 且满足 $L+2 \leq r \leq 2$ 的项比经典项奇异性更强。

2. The classical piece of order $1/\hbar$ from which the classical Einstein gravity contribution is extracted. It showed the presence of a classical piece in the quantum gravitational two-body amplitude at one loop [5], as articulated as an all order statement in [115]. The expansion in (37) is unusual, but this is a natural when considering a huge external mass expansion of the two-body gravitational scattering. At the $L+1$ post-Minkowskian order, the two-body scattering amplitude between two massive particles has the following mass dependence:

2. 阶为 $1/\hbar$ 的经典项，经典爱因斯坦引力贡献即从中提取。已有研究指出，单圈量子引力双体振幅中存在经典项 [5]，文献 [115] 将其表述为全阶结论。(37) 式中的展开并不常见，但在考虑双体引力散射的大外质量展开时，这是自然的结果。在 $L+1$ 后闵可夫斯基阶下，两个有质量粒子间的双体散射振幅满足如下质量依赖关系:

$$\mathcal{M}_L\left(\gamma, q^2\right) = \frac{G_N^{L+1} m_1^2 m_2^2}{\underline{q}^{2+\frac{(2-D)L}{2}}} \sum_{i=0}^{L} c_{L-i+2, i+2}(\gamma)\, m_1^{L-i} m_2^i. \tag{38}$$

This classical contribution emerges from the $1/\hbar$ piece of the quantum amplitude in (37) remembering that the mass dependence in quantum field theory appears as the Compton wavelength $mc/\hbar$. Expressing the classical contribution by making this explicit gives

该经典贡献来自 (37) 式中量子振幅的 $1/\hbar$ 项，我们需要注意，量子场论中的质量依赖以康普顿波长 $mc/\hbar$ 的形式出现。将该关系明确写出后，经典贡献可表示为

$$\mathcal{M}_L\left(\gamma, q^2, \hbar\right) = \cdots + \frac{\frac{m_1^2 m_2^2}{q^{2+\frac{(2-D)L}{2}}} \hbar^{L-1} G_N^{L+1} \sum_i \left(\frac{m_1 c}{\hbar}\right)^{L-i} \left(\frac{m_2 c}{\hbar}\right)^i}{\frac{\mathcal{M}_L(\gamma, q^2)}{\hbar}} + \cdots$$

(39)

Therefore, the polynomial mass dependence of the classical amplitude in (38) expected for the conservative part of the scattering angle [14, 117] arises consistently from the classical limit of the quantum amplitude. The q^2 dependence of the classical contributions is exactly what one anticipates to contribute to the three-dimensional potential at the $L+1$ post-Newtonian order since

因此，散射角保守部分 [14, 117] 所预期的 (38) 式中经典振幅的多项式质量依赖，与量子振幅经典极限的结果一致。经典贡献对 q^2 的依赖，完全符合人们对 $L+1$ 后牛顿阶下三维势贡献的预期，因为

$$G_N^{L+1} \int d^3\underline{\mathbf{q}} \frac{e^{i\underline{q}\cdot\mathbf{r}}}{\left|\underline{q}\right|^{2-L}} \propto \left(\frac{G_N}{r}\right)^{L+1}. \tag{40}$$

The expansion in (37) indicates that a given $\hbar$ order has a prescribed analytic dependence in q^2 . We utilize this to elucidate the classical contribution from specific unitarity cuts [10]. The extraction of the classical part has been since systematized using a heavy-mass effective theory approach [25, 118] or the velocity cut formalism [22, 23, 29] which is discussed further in section ”New Computational Technology for Gravitational Scattering Amplitudes”.

(37) 式的展开表明，任意给定 $\hbar$ 阶，其对 q^2 都有确定的解析依赖。我们利用这一点来阐明特定幺正切割得到的经典贡献 [10]。此后，经典项提取已经通过重质量有效理论方法 [25, 118] 或速度切割形式 [22, 23, 29] 系统化，我们会在“引力散射振幅的新型计算技术”一节中进一步讨论后者。

3. The quantum corrections of order $\hbar^r$ (with $r \geq 0$) lead to quantum gravity corrections to the classical Einstein gravity results [82].

3. 阶为 $\hbar^r$ 的量子修正 (满足 $r \geq 0$) 会给经典爱因斯坦引力结果带来量子引力修正 [82]。

Scattering Potentials from the Lippmann-Schwinger Equation and Gravitational Eikonalization

由李普曼-施温格方程和引力程函化得到散射势

Supplied with the scattering amplitude, we can relate to a Hamiltonian scattering potential via numerous avenues; for instance, one can derive a classical Hamiltonian by employing Born subtractions in the framework of the Lippmann-Schwinger equation [12, 119, 120] . In this setup, one links the relativistic scattering amplitude $\mathcal{M}$ to the potential $\mathcal{V}$, through

已知散射振幅后，我们可以通过多种途径将其与哈密顿散射势联系起来；例如，可以在李普曼-施温格方程的框架内利用玻恩减法推导经典哈密顿量 [12, 119, 120] 。在该设置下，人们将相对论散射振幅 $\mathcal{M}$ 与势 $\mathcal{V}$ 通过下式关联

$$\widetilde{\mathcal{M}}_L(p, p') \equiv \frac{\mathcal{M}_L(p, p')}{4E_1E_2}, \tag{41}$$

with

其中

$$p_1 = (E_1, \mathbf{p}),\ p_1' = (E_1, \mathbf{p}'),\ p_2 = (E_2, -\mathbf{p}),\ p_2' = (E_2, -\mathbf{p}'), \tag{42}$$

$$\mathcal{V}(r, p) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} \mathcal{V}(p, q) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} \widetilde{\mathcal{M}}_0(p, q). \tag{43}$$

$$\widetilde{\mathcal{M}}(p, p') = \mathcal{V}(p, p') + \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{V}(p, k)\mathcal{M}(k, p')}{E_p - E_k + i\varepsilon}. \tag{44}$$

This extension, through a one-particle Salpeter equation, explains - simply and systematically - the subtractions mandated, in principle to any perturbative order. However, at high perturbative loop order, it is an arduous task to keep track of the multiple new computational pieces concerned (at this point in writing, only a careful examination of two-loop order exists). An alternative framework for determining a Hamiltonian is using the effective field theory matching strategy of [11, 13, 15, 26, 27, 121] . One can ascertain the parallelism between these two systems to hold on general grounds [122].

这种通过单粒子萨尔佩特方程实现的推广，简单且系统地解释了原则上对任意微扰阶都要求的减法。但在高微扰圈阶下，追踪涉及的多个新增计算项是一项艰巨的任务 (截至撰写本文时，仅对两圈阶完成了细致研究)。确定哈密顿量的另一个框架是采用 [11, 13, 15, 26, 27, 121] 的有效场论匹配策略。可以确认，这两种框架的等效性普遍成立 [122]。

The recourse to a Hamiltonian framework for computing observables is the eikonal formalism. Employing the eikonal formalism, one converts the amplitude following

程函形式体系是哈密顿框架用于计算可观测量的一种应用。利用程函形式体系，人们按下式变换振幅

$$\mathcal{M}_L(\gamma, b, \hbar) = \frac{1}{4m_1m_2\sqrt{\gamma^2 - 1}} \int_{\mathbb{R}^{D-2}} \frac{d^{D-2}\underline{\mathbf{q}}}{(2\pi)^{D-2}} \mathcal{M}_L\left(\gamma, \underline{q}^2, \hbar\right) e^{i\underline{\mathbf{q}}\cdot\mathbf{b}}, \tag{45}$$

from momentum transfer space to impact parameter space. The classical eikonal phase $\delta(\gamma, b)$ is associated with an exponentiation of the S -matrix in b -space. The effect is a representation of the amplitude in terms that exponentiate with the eikonal phase $\delta(\gamma, b)$ and contributions that do not and that one has to incorporate in $\Delta(\gamma, b, \hbar)$

从动量转移空间变换到碰撞参数空间。经典程函相位 $\delta(\gamma, b)$ 对应于 b 空间中 S 矩阵的指数化。其结果是将振幅表示为含程涵相位 $\delta(\gamma, b)$ 的指数项，以及非指数项，这些非指数项必须纳入 $\Delta(\gamma, b, \hbar)$

$$1 + i\mathcal{T} = (1 + i2\Delta(\gamma, b, \hbar))\, e^{\frac{2i\delta(\gamma,b)}{\hbar}}. \tag{46}$$

The exponentiation of the amplitude in impact parameter space is a consequence of the unitarity relations of the S -matrix [122], and it has an analog to the method of Born subtractions.

碰撞参数空间中振幅的指数化是 S 矩阵幺正性关系的结果 [122]，它与玻恩减法方法存在对应关系。

Furnished with amplitudes, we can regard the phase $\delta(\gamma, b)$ as independent of quantum corrections. It is conventional to expand it in powers of G_N through perturbative orders where scattering amplitudes are expanded as $1 + i\mathcal{T} = 1 + \ i \sum_{L\geq 0} \mathcal{M}_L(\gamma, b, \hbar)$; thus, we have

已知振幅后，我们可以将相位 $\delta(\gamma, b)$ 视为与量子修正无关。按惯例将其按 G_N 的幂次展开，对应散射振幅按 $1 + i\mathcal{T} = 1 + \ i \sum_{L\geq 0} \mathcal{M}_L(\gamma, b, \hbar)$ 展开的各微扰阶，因此我们有

$$\delta(\gamma, b) = \sum_{L\geq 0} \delta_L(\gamma, b)\,. \tag{47}$$

By solving saddle point conditions, we can use the eikonal phases carefully extracted from amplitudes to assess observables such as the scattering angle. Utilizing this, we can instantly infer at the tree-level and one-loop level,

通过求解鞍点条件，我们可以利用从振幅中仔细提取的程函相位来计算散射角这类可观测量。利用该方法，我们可以直接得到树图阶和单圈阶的结果为

$$\delta_0(\gamma, b) = G_N m_1 m_2 \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}} \frac{\left(\pi b^2 e^{\gamma_E}\right)^{\frac{4-D}{2}}}{D - 4} + \mathcal{O}\left((D-4)^0\right), \tag{48}$$

$$\delta_1(\gamma, b) = G_N^2 (m_1 + m_2)\, m_1 m_2 \frac{3\pi\left(5\gamma^2 - 1\right)}{8b\sqrt{\gamma^2 - 1}} \left(\pi b^2 e^{\gamma_E}\right)^{4-D} + \mathcal{O}(4 - D), \tag{49}$$

and if we take care of iterations at two-loop order,

如果我们处理好两圈阶的迭代，可得

$$\begin{aligned}
\delta_2(\gamma, b) = {} & \frac{G_N^3 m_1 m_2 \left(\pi b^2 e^{\gamma_E}\right)^{\frac{3(4-D)}{2}}}{2b^2\sqrt{\gamma^2 - 1}} \left(\frac{2\left(12\gamma^4 - 10\gamma^2 + 1\right)\mathcal{E}_{\text{C.M.}}^2}{\gamma^2 - 1}\right. \\
& - \frac{4m_1 m_2 \gamma}{3}\left(25 + 14\gamma^2\right) + \frac{4m_1 m_2\left(3 + 12\gamma^2 - 4\gamma^4\right)\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \\
& \left. + \frac{2m_1 m_2 \left(2\gamma^2 - 1\right)^2}{\sqrt{\gamma^2 - 1}} \frac{1}{\left(4\left(\gamma^2 - 1\right)\right)^{\frac{4-D}{2}}} \left(-\frac{11}{3} + \frac{d}{d\gamma}\left(\frac{\left(2\gamma^2 - 1\right)\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}}\right)\right)\right) \\
& + \mathcal{O}(4 - D)
\end{aligned} \tag{50}$$

These results agree with Ref. [17].

这些结果与文献 [17] 一致。

New Computational Technology for Gravitational Scattering Amplitudes

引力散射振幅的新型计算技术

We have observed that we can compute scattering amplitudes from unitarity cuts using on-shell compact expressions for tree amplitudes. It is advantageous to adjust the computational method and adapt computations according to a large mass expansion to streamline such computations concentrated on extracting the classical part from the full quantum amplitude. One can accomplish this by pursuing two routes. The first is a transformation of the heavy-mass effective theory approach [25, 118], and the second is adjusting the unitarity method for determining the mass contributions from massive cuts. This latter technique, dubbed as the velocity cut method in [22, 23, 29] , we will demonstrate here. Such methods derive a refined computational path that concentrates on the integrand components in the amplitude that deliver the classical radial action.

我们已经发现，利用树图振幅的在壳紧致表达式，我们可以从幺正切割计算散射振幅。调整计算方法、根据大质量展开适配计算是有利的，这可以精简这类专注于从全量子振幅中提取经典部分的计算。这可以通过两种路径实现。第一种是对重质量有效理论方法的改造 [25, 118], 第二种是调整幺正方法来确定来自有质量切割的质量贡献。后一种技术被命名为速度切割方法，我们将在 [22, 23, 29] 中展示它。这类方法提供了一条更精细的计算路径，专注于振幅中给出经典径向作用量的被积函数分量。

The starting point for this refined examination is extracting the radial action from an exponentiated S -matrix. As we have witnessed in the previous section, one can constrain exponential phases δ in impact parameter by the unitarity of the S -matrix, thus playing a critical role in deriving the classical Einstein gravity from amplitudes. It motivates the following framework for computations [24] where one develops an exponential representation of the S -matrix at the operator level in momentum transfer space to generate the radial action directly. We will subsequently review this formalism as it plays an essential role in the simplified computational framework that we will evolve later in the section. We begin with

这项精细研究的出发点是从指数化的 S 矩阵中提取径向作用量。正如我们在前一节中看到的，冲击参数中的指数相位 δ 可以通过 S 矩阵的幺正性得到约束，因此它在从振幅推导经典爱因斯坦引力中发挥关键作用。这催生了如下计算框架 [24]: 在动量转移空间的算符层级构建 S 矩阵的指数表示，以直接生成径向作用量。我们随后将回顾这一形式体系，因为它对我们本节后面将发展的简化计算框架至关重要。我们从下式开始

$$\hat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right), \tag{51}$$

where the scattering operator $\widehat{T}$ and the $\widehat{N}$ operator (which is associated with the radial action) have the expansion

其中散射算符 $\widehat{T}$ 和 $\widehat{N}$ 算符 (与径向作用量相关联) 满足展开式

$$\widehat{T} = \sum_{n\geq 0} G_N^n \widehat{T}_n + \sum_{n\geq 0} G_N^{n+\frac{1}{2}} \widehat{T}_n^{\mathrm{rad}};\ \widehat{N} = \sum_{n\geq 0} G_N^n \widehat{N}_n + \sum_{n\geq 0} G_N^{n+\frac{1}{2}} \widehat{N}_n^{\mathrm{rad}}; \tag{52}$$

together with the completeness relation

以及完备关系

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{2} \frac{d^{D-1}k_i}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{k_1}} \prod_{j=1}^{n} \frac{d^{D-1}\ell_j}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_j}} |k_1, k_2; \ell_1, \dots \ell_n\rangle \langle k_1, k_2; \ell_1, \dots \ell_n|,$$

(53)

which encloses all the exchanges of gravitons for $n \geq 1$ entering the radiation-reaction contributions $\widehat{T}^{\mathrm{rad}}$ and $\widehat{N}^{\mathrm{rad}}$. With this exponential representation of the S-matrix, we can systematically correlate matrix elements of the operator in the exponential $\widehat{N}$ to Born amplitudes minus components supplied by unitarity cuts [24]. The perturbation expansion demonstrates it,

它包含了进入辐射反作用贡献 $\widehat{T}^{\mathrm{rad}}$ 和 $\widehat{N}^{\mathrm{rad}}$ 的所有 $n \geq 1$ 引力子交换。借助 S 矩阵的这种指数表示，我们可以系统地将指数 $\widehat{N}$ 中算符的矩阵元关联到由幺正切割提供的玻恩振幅减去分量 [24]。微扰展开证明了这一点,

$$\widehat{N}_0 = \widehat{T}_0,\ \widehat{N}_0^{\mathrm{rad}} = \widehat{T}_0^{\mathrm{rad}}$$

$$\widehat{N}_1 = \widehat{T}_1 - \frac{i}{2\hbar}\widehat{T}_0^2,\ \widehat{N}_1^{\mathrm{rad}} = \widehat{T}_1^{\mathrm{rad}} - \frac{i}{2\hbar}\left(\widehat{T}_0\widehat{T}_0^{\mathrm{rad}} + \widehat{T}_0^{\mathrm{rad}}\widehat{T}_0\right),$$

$$\widehat{N}_2 = \widehat{T}_2 - \frac{i}{2\hbar}\left(\widehat{T}_0^{\mathrm{rad}}\right)^2 - \frac{i}{2\hbar}\left(\widehat{T}_0\widehat{T}_1 + \widehat{T}_1\widehat{T}_0\right) - \frac{1}{3\hbar^2}\widehat{T}_0^3, \tag{54}$$

and similarly for higher orders.

更高阶的情况与此类似。

From the higher-order terms in the $\hbar$ expansion (54), one can as well study the quantum gravity corrections [65, 82, 112, 123 – 125] to the leading classical contributions and appeal to unitarity and analyticity arguments [126] for constraining some higher-derivative terms in the effective action (4). The next step for the exploration is multi-soft expansions of the tree-level amplitudes that enter unitarity cuts in the integrand construction. First, we determine a basis of integral functions that enter a given unitarity cut. Next, we consider the constraints levied by unitarity on the integrand. We can, for instance, at L-loop order, consider a generalized $(L + 1)$ graviton cut (The multi-graviton cut is not enough to reconstruct the full classical L-loop amplitude. At two loops, we, for instance, require to add the bowtie diagram where one factorizes the amplitude into a product of two scalar-graviton amplitudes times graviton amplitude. Another type of graph we must add is a self-energy graph.).

从 (54) 式 $\hbar$ 展开的高阶项中，我们还可以研究领头经典贡献的量子引力修正 [65, 82, 112, 123 − 125]，并借助幺正性和解析性论证 [126] 来约束有效作用量 (4) 中的一些高阶导数项。后续研究的下一步是进入被积函数构造幺正切割的树图振幅多软展开。首先，我们确定进入给定幺正切割的积分函数基。接着，我们考虑幺正性对被积函数施加的约束。例如，我们可以在 L 圈阶考虑广义 $(L+1)$ 引力子切割 (多引力子切割不足以重构完整的经典 L 圈振幅。例如在两圈阶，我们需要添加领结图，其中我们将振幅因子化为两个标量-引力子振幅乘以引力子振幅的乘积。我们必须添加的另一类图是自能图。)。

$$i\mathcal{M}_L^{\text{cut}}\left(\sigma, q^2\right) = \hbar^{3L+1} \int (2\pi)^D \delta\left(q + \ell_2 + \cdots + \ell_{L+2}\right) \prod_{i=2}^{L+2} \frac{i}{\ell_i^2} \prod_{i=2}^{L+2} \frac{d^D \ell_i}{(2\hbar\pi)^D}$$

$$\frac{1}{(L+1)!} \sum_{h_i=\pm 2} M_{\text{Left}}^{\text{tree}}\left(p_1, \ell_2^{h_2}, \ldots, \ell_{L+2}^{h_{L+2}}, -p_1'\right) M_{\text{Right}}^{\text{tree}}\left(p_2, -\ell_2^{h_2}, \ldots, -\ell_{L+2}^{h_{L+2}}, -p_2'\right)^{\dagger}.$$

(55)

Here, $M_{\text{Left}}^{\text{tree}}\left(p_1, \ell_2, \ldots, \ell_{L+2}, -p_1'\right)$ and $M_{\text{Right}}^{\text{tree}}\left(p_2, -\ell_2, \ldots, -\ell_{L+2}, -p_2'\right)$ are tree-level amplitudes corresponding to multi-graviton emission from a massive scalar line. Now in all outgoing line conventions, we can write the following equation for momentum conservation:

此处，$M_{\text{Left}}^{\text{tree}}\left(p_1, \ell_2, \ldots, \ell_{L+2}, -p_1'\right)$ 和 $M_{\text{Right}}^{\text{tree}}\left(p_2, -\ell_2, \ldots, -\ell_{L+2}, -p_2'\right)$ 是对应于有质量标量线辐射多个引力子的树图振幅。现在在所有出射线约定下，我们可以写出动量守恒的方程如下:

$$q = p_1 - p_1' = -\sum_{i=2}^{L+2} \ell_i \tag{56}$$

Now, an important observation is that if we consider a tree amplitude as

现在，一个重要的结论是: 如果我们将树图振幅视为

$$M_{\text{Left}}^{\text{tree}}\left(p_1, \ell_2, \ldots, \hat{\ell}_i, \ldots, \ell_{L+1}, \ell_{L+2}, -p_1'\right), \tag{57}$$

where we use momentum conservation

这里我们利用了动量守恒

$$\ell_i = -q - \sum_{\substack{2 \le j \le L+2 \\ j \ne i}} \ell_j. \tag{58}$$

One observes that in (57), different types of massive propagators exist. We have propagators with a ”hatted” momentum for which we can write

不难发现，在 (57) 式中存在不同类型的有质量传播子。带有“带帽”动量的传播子可以写为

$$\frac{1}{\left(p_1 - \ell_{i_2} - \cdots - \ell_{i_j} - q\right)^2 - m^2 + i\varepsilon}$$

$$= \frac{-1}{2p_1 \cdot \left(\ell_{i_2} + \cdots + \ell_{i_j}\right) - \left(\ell_{i_2} + \cdots + \ell_{i_j} + q\right)^2 - i\varepsilon}, \tag{59}$$

and propagators without a "hatted" momentum

不带“带帽”动量的传播子写为

$$\frac{1}{\left(p_1 + \ell_{i_2} + \cdots + \ell_{i_j}\right)^2 - m^2 + i\varepsilon} = \frac{1}{2p_1 \cdot \left(\ell_{i_2} + \cdots + \ell_{i_j}\right) + \left(\ell_{i_2} + \cdots + \ell_{i_j}\right)^2 + i\varepsilon}.$$

(60)

(We have taken $1 < j \leq L + 2$.)

(我们已取 $1 < j \leq L + 2$ 。)

Now, the critical observation is the following. If we employ the equation

现在，关键结论如下。如果我们利用方程

$$\lim_{\varepsilon \to 0^+} \left(\frac{1}{\eta - i\varepsilon} - \frac{1}{\eta + i\varepsilon} \right) = \lim_{\varepsilon \to 0^+} \frac{2i\varepsilon}{\eta^2 + \varepsilon^2} = 2i\pi\delta\left(\eta\right) \tag{61}$$

and recast propagators with a "hat,"

改写带“帽”的传播子,

$$\frac{1}{\left(p_1 - \ell_{i_2} - \cdots - \ell_{i_j} - q\right)^2 - m^2 + i\varepsilon} = -2i\pi\delta\left(\left(p_1 - \ell_{i_2} - \cdots - \ell_{i_j} - q\right)^2 - m^2\right)$$

$$+ \frac{1}{\left(p_1 - \ell_{i_2} - \cdots - \ell_{i_j} - q\right)^2 - m^2 - i\varepsilon}.$$

(62)

One deduces a new organizing principle for integrands. Iterating applications of this identity, one can derive a form tree amplitudes with two scalar legs and $L + 1$ graviton legs organized with different powers of unitarity cuts delta functions. Defining $M_{L+1}^{\text{tree}\,(+)}(p_1, \ell_2, \ldots, \ell_{L+2}, -p_1')$ as the tree-level amplitudes with propagators $2p_1 \cdot \left(\sum_r \ell_r\right) - \left(q + \sum_r \ell_r\right)^2 - i\varepsilon$ rewritten as $2p_1 \cdot \left(\sum_r \ell_r\right) - \left(q + \sum_r \ell_r\right)^2 + i\varepsilon$ and similarly $M_{L+1}^{\text{tree}\,(-)}(p_1, \ell_2, \ldots, \ell_{L+2}, -p_1')$ with propagators $2p_1 \cdot \left(\sum_r \ell_r\right) + \left(\sum_r \ell_r\right)^2 + i\varepsilon$ changed to $2p_1 \cdot \left(\sum_r \ell_r\right) + \left(\sum_r \ell_r\right)^2 - i\varepsilon$, we arrive at

我们推导出被积函数的全新组织原理。通过反复应用该恒等式，可以得到具有两个标量腿和 $L+1$ 个引力子腿的树振幅，该振幅按幺正切割 δ 函数的不同幂次组织。将 $M_{L+1}^{\text{tree}\,(+)}(p_1, \ell_2, \ldots, \ell_{L+2}, -p_1')$ 定义为传播子 $2p_1 \cdot \left(\sum_r \ell_r\right) - \left(q + \sum_r \ell_r\right)^2 - i\varepsilon$ 改写为 $2p_1 \cdot \left(\sum_r \ell_r\right) - \left(q + \sum_r \ell_r\right)^2 + i\varepsilon$ 后的树级振幅，同理将 $M_{L+1}^{\text{tree}\,(-)}(p_1, \ell_2, \ldots, \ell_{L+2}, -p_1')$ 定义为传播子 $2p_1 \cdot \left(\sum_r \ell_r\right) + \left(\sum_r \ell_r\right)^2 + i\varepsilon$ 改写为 $2p_1 \cdot \left(\sum_r \ell_r\right) + \left(\sum_r \ell_r\right)^2 - i\varepsilon$ 后的树级振幅，我们得到

$$M_{L+1}^{\text{tree}} \sim \left(M_1^{\text{tree}\,(+)}\right)^{L+1} \prod_i^L \delta_i(\ldots) + \left(M_1^{\text{tree}\,(+)}\right)^{L-1} \left(M_2^{\text{tree}\,(+)}\right) \prod_i^{L-1} \delta_i(\ldots) + \cdots$$
$$+ M_1^{\text{tree}\,(+)} M_L^{\text{tree}\,(+)} \delta(\ldots) + M_{L+1}^{\text{tree}\,(+)}, \tag{63}$$

as a new representation of the tree-level amplitude. The important observation is that it is now relatively easy to read the soft behavior of such trees in unitarity cuts since, considering the soft scaling behavior, we have

作为树级振幅的一种新表示形式。一个重要的结论是，现在我们可以很容易地分析此类树振幅在幺正切割下的软行为，因为考虑软标度行为后，我们有

$$\delta\left(\left(p_1 + \sum \ell_i\right)^2 - m_1^2\right) = \delta\left(2\,|q|\, p_1 \cdot \sum \tilde{\ell}_i + \mathcal{O}\left(|q|^2\right)\right)$$
$$= \frac{1}{|q|} \delta\left(2p_1 \cdot \sum \tilde{\ell}_i\right) + \mathcal{O}\left(|q|^0\right), \tag{64}$$

and thus the amplitude $\mathcal{M}_{L+1}$ has multi-soft scaling

因此振幅 $\mathcal{M}_{L+1}$ 满足多软标度关系

$$\lim_{|\mathbf{q}| \to 0} M_{L+1}^{\text{tree}}\left(p, |\mathbf{q}|\, \tilde{\ell}_2, \ldots, |\mathbf{q}|\, \hat{\tilde{\ell}}_{L+2}, -p'\right) = \frac{\left(M_1^{\text{tree}}\right)^{L+1} \delta(\cdots)^L}{|\mathbf{q}|^L} + \mathcal{O}\left(\frac{1}{|q|^{L-1}}\right).$$

(65)

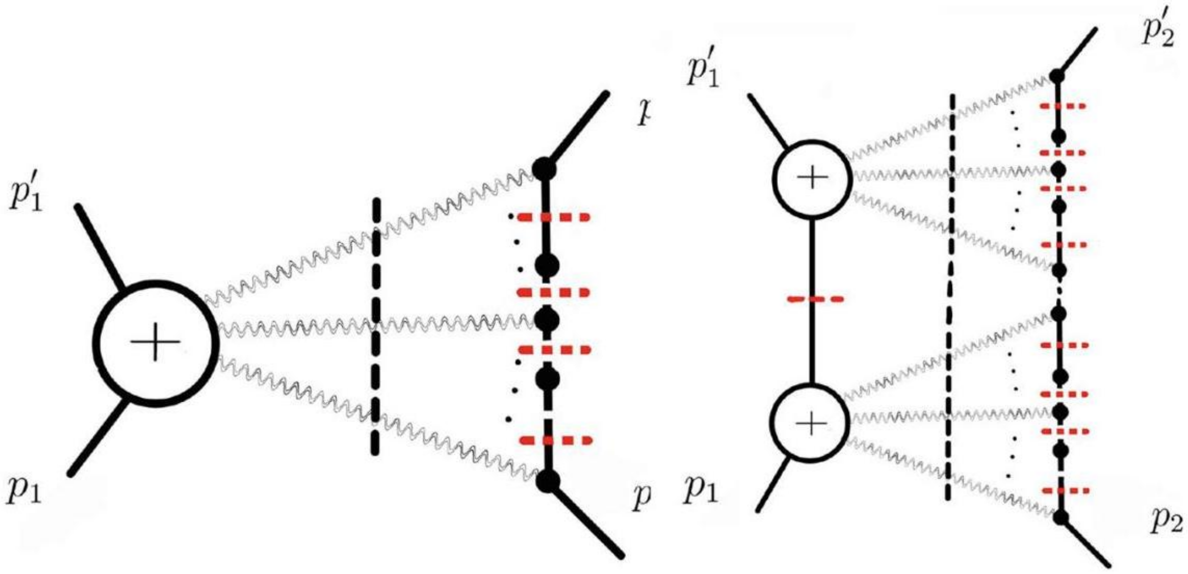


Fig. 6 The probe (left) and next-to-probe contribution (right)

图 6 探测项贡献 (左) 和次探测项贡献 (右)

Using this decomposition in the integrand derived from generalized unitarity and combining this knowledge with the exponential representation of the S -matrix provided in Ref. [24], one arrives simply with an integrand of the classical radial action. It was in Ref. [29] developed as a new practical technique for calculating post-Minkowskian dynamics.

将该分解应用于从广义幺正性导出的被积函数，并结合参考文献 [24] 给出的 S 矩阵的指数表示，我们可以直接得到经典径向作用量的被积函数。该方法在参考文献 [29] 中被发展为一种计算后闵可夫斯基动力学的新型实用技术。

For instance, focusing on the leading probe contribution, we see that it arises the following type of contribution in the cut:

例如，针对领头探测项贡献，我们可以看到它在切割中产生如下类型的贡献:

$$\left(M_{\text{Left}} M_{\text{Right}}{}^{\dagger}\right)\Big|_{\text{probe}} = M_{L+1}^{\text{tree}\,(-)\dagger}\left(p_2, -\hat{\ell}_2, -\ell_3, \ldots, -\ell_{L+1}, -p_2'\right)$$

$$\times M_1^{\text{tree}(+)}\left(p_1, \hat{\ell}_2, -p_1 - \hat{\ell}_2\right) \prod_{j=3}^{L} \delta\left(\left(p_1 + \hat{\ell}_2 + \cdots + \ell_{j-1}\right)^2 - m_1^2\right)$$

$$M_1^{\text{tree}\,(+)}\left(p_1 + \hat{\ell}_2 + \cdots + \ell_{j-1}, \ell_j, -p_1 - \hat{\ell}_2 - \cdots + \ell_j\right), \tag{66}$$

where the cut is evaluated using $\ell_i^2 = 0$ for $1 \le i \le L+1$. We show this contribution in the left Fig. 6. Contributions also arise from next-to-probe topologies which is represented in the right Fig. 6.

其中切割使用 $\ell_i^2 = 0$ 对 $1 \leq i \leq L+1$ 计算，我们将该贡献示于图 6 左图。次探测拓扑也会产生贡献，这类贡献示于图 6 右图。

Fourth-Order Post-Minkowskian Dynamics

四阶后闵可夫斯基动力学

At the fourth post-Minkowskian order, new puzzles arise regarding the derivation of gravitational radiation, and this presents a strong test for validating the understanding of the logic developed for deriving the post-Minkowskian expansion in classical gravity.

在四阶后闵可夫斯基近似下，引力辐射推导出现了新的谜题，这也为检验经典引力中后闵可夫斯基展开推导逻辑的正确性提供了有力的测试。

For this perturbation order, we can split contributions into conservative and radiation pieces; the conservative has been derived in [26-28,53,54,127,128].

对于该微扰阶数，我们可将贡献拆分为保守项和辐射项；其中保守项已在文献 [26-28,53,54,127,128] 中推导得到。

At the fourth post-Minkowskian order, a complete computation of the conservative and radiative sector has been presented recently in [54]. In this section, we explain the amplitude-based approach for the conservative part at this order. This is an application of the velocity cut formalism, presented in the previous section. We explain that this formalism leads to the re-derivation of the conservative piece from a small subset of maximal cut topologies. The following discussion does not include all the gravitational radiation effects which need a separate analysis in the scattering amplitude setup.

近期文献 [54] 报道了四阶后闵可夫斯基近似下保守项与辐射项的完整计算。本节我们将介绍该阶数下基于振幅的保守项求解方法，这是上一节介绍的速度切割形式化方法的一个应用。我们将说明，该形式化方法可以仅通过一小部分最大切割拓扑重新推导出保守项。后续讨论不包含全部引力辐射效应，这类效应需要在散射振幅框架下单独分析。

The real part of the classical two-body gravitational interaction from the three-loop scattering has the mass expansion

三圈散射给出的经典两体引力相互作用实部具有如下质量展开式

$$\mathcal{M}_{4\mathrm{PM}}\left(\gamma, \underline{q}^2\right)\Big|_{\text{classical}} = \lim_{\varepsilon\to 0}\left(\frac{G_N^4 m_1^2 m_2^2}{\left|\underline{q}\right|^{-1+3\varepsilon}}(m_1+m_2)\left((m_1+m_2)^2 c^{\,\text{probe}}(\gamma,\varepsilon)\right.\right.$$
$$\left.\left. + m_1 m_2 c^{\,\text{next-to-probe}}(\gamma,\varepsilon)\right)\right)\ . \tag{67}$$

We evaluate the amplitude in $D = 4-2\varepsilon$, to control the infrared divergences from gravitational radiation. The expression has a finite four-dimensional limit, as expected. This result has the generic polynomial mass dependence from (38) with $L = 4$ due to the delta-function count for obtaining the classical piece.

我们在 $D = 4 - 2\varepsilon$ 中计算振幅，以控制引力辐射带来的红外发散。正如预期，该表达式在四维极限下有限。根据得到经典项的 δ 函数计数规则，该结果符合式 (38) 给出的通用多项式质量依赖关系，其中系数为 $L = 4$ 。

Taking the classical limit $\hbar \to 0$ imposes delta-function insertions' massive lines, as explained in the previous section. We have represented them with a red line in Fig. 7. These delta-function insertions cut open the massive lines and lead to the important reduction of the integrand at the fourth post-Minkowskian order as a sum over permutations of planar tree graphs listed in Fig. 8.

如前所述，取经典极限 $\hbar \to 0$ 时需要在质量线中插入 δ 函数。我们在图 7 中用红线标记了这些插入。δ 函数插入将质量线切开，使四阶后闵可夫斯基近似下的被积函数可以约化为图 8 中列出的平面树图排列和。

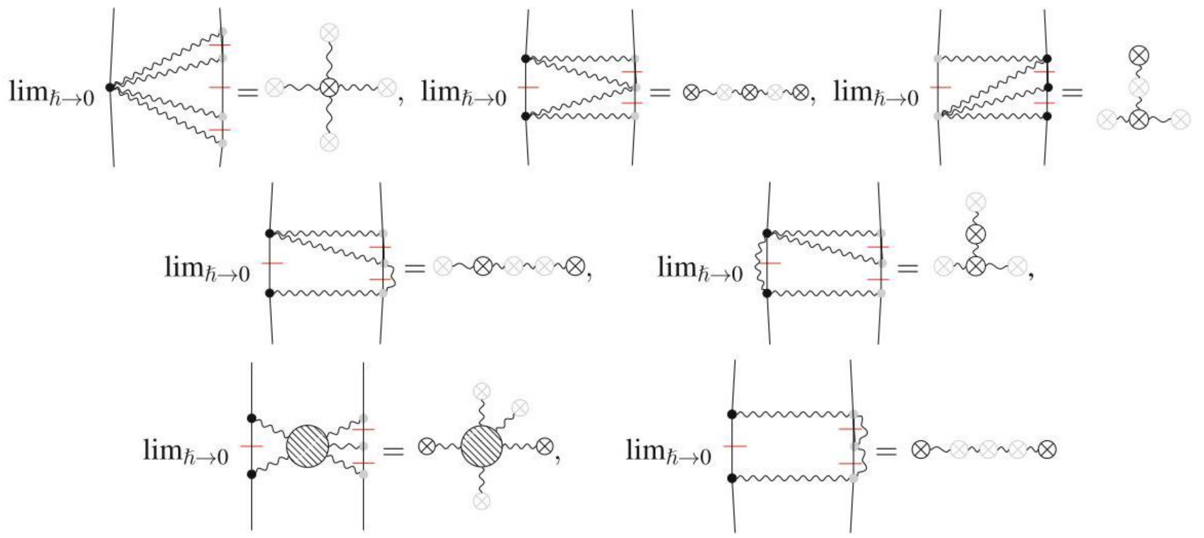


Fig. 7 The classical limit of the velocity cuts

图 7 速度切割的经典极限

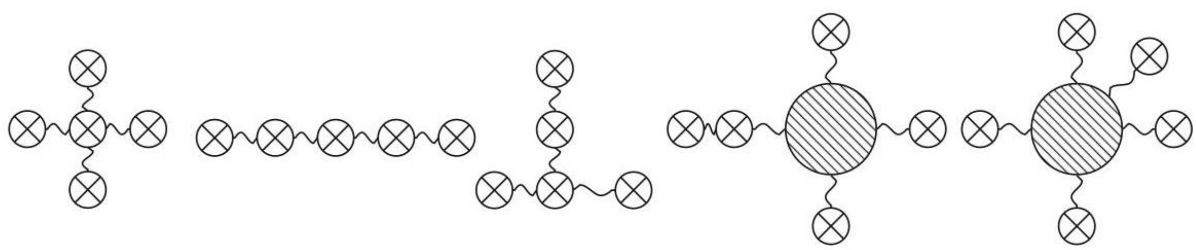


Fig. 8 Only four tree topologies are needed to reconstruct the full 4 PM integrand. We denote sources by crossed dots in the graphs; the dashed blob is a multi-graviton amplitude

图 8 重构完整四阶后闵可夫斯基被积函数仅需要四种树拓扑。图中叉点表示源；虚线 blob 是多引力子振幅

In assembling the integrand, it is a fact that non-planar integrand topologies can appear [26-28, 53, 128]. However, because of the velocity cuts, we have found that it is always possible to assemble these contributions by use of a partial fractioning identity that maps the non-planar integral denominator into a combination of planar denominators. It uses that when the scalar line is cut open, permutations of the graviton lines, attached to the scalar lines, are mapped to the same type tree-level graph topologies above, modulo, a permutation of the assignment of legs. One can thus rephrase the non-planar contributions from the multi-graviton tree-level insertions in terms of planar, and thus, as a consequence, the reduction of master integrals needs to only involve planar graphs. We also need to eliminate overlapping cut topology contributions. They are removed directly in our reconstruction of the integrand by introducing symmetry factors in certain integrand sectors. For instance, the listed topologies account for the contribution with a four-graviton amplitude, as displayed in the figure below

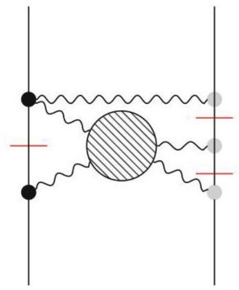
在组装被积函数时，非平面被积函数拓扑确实会出现 [26-28, 53, 128]。但由于速度切割，我们发现总可以通过部分分式恒等式组装这些贡献，将非平面积分分母改写为平面分母的组合。该恒等式成立的原因是: 当标量线被切开后，附着在标量线上的引力子线排列，除外腿分配排列不同外，都会映射到上述同类型树图拓扑。因此我们可以将多引力子树插入带来的非平面贡献改写为平面贡献，结果主积分约化仅需要处理平面图形即可。我们还需要消除重叠切割拓扑的贡献，在我们的被积函数重构中，直接通过给部分被积扇区引入对称因子就可以去除这些贡献。例如，下图所示的四引力子振幅贡献已经包含在列出的拓扑中

$$\mathcal{M}_{4\mathrm{PM}}\left(\gamma,\underline{q}^2\right)=\lim_{\varepsilon\to 0}\sum_{i=1}^{40} c\left(\{n_j\};\gamma,\underline{q}^2\right)\mathcal{I}\left(\{n_j\};\gamma,\varepsilon\right), \tag{68}$$

with the family of master integrals evaluated in $D=4-2\varepsilon$ dimensions

其中主积分族在 $D=4-2\varepsilon$ 维下计算得到

$$\mathcal{I}\left(\{n_j\};\gamma,\varepsilon\right)=\int\frac{1}{\prod_{i=1}^{12}D_i^{n_i}}\prod_{r=1}^{3}\frac{2\pi i\delta\left(2u_1\cdot\sum_{j=1}^{r}l_j\right)d^{4-2\varepsilon}l_i}{(2\pi)^{4-2\varepsilon}}, \tag{69}$$



Performing the tensor reduction with LiteRed [129], the 4th post-Minkowskian radial action is expandable on 40 master integrals (For the conservative sector, the world-line approach of [54] has up to 576 master integrals because of the presence of retarded or advanced propagators in their formalism.)

利用 LiteRed 做张量约化 [129] 后，四阶后闵可夫斯基径向作用量可以展开为 40 个主积分 (对于保守扇区，文献 [54] 的世界线方法最多需要 576 个主积分，因为他们的形式化方法中包含推迟或超前传播子。)

where we have defined the propagators

其中我们对传播子做如下定义

$$D_1=(l_1+l_2+q)^2, D_2=(l_1+l_2)^2, D_3=2u_1\cdot(l_1+l_2), D_4=l_2^2, D_5=l_3^2,$$

$$D_6=l_1^2, D_7=(l_1+l_2+l_3+q)^2, D_8=(l_1+q)^2, D_9=(l_1+l_2+l_3)^2,$$

$$D_{10}=-2u_2\cdot l_1, D_{11}=-2u_2\cdot(l_1+l_2+l_3), D_{12}=(l_2+l_3)^2, \tag{70}$$

with $u_1^2=u_2^2=1, \gamma=u_1\cdot u_2, q^2=-1$, and $u_1\cdot q=u_2\cdot q=0$. It is useful to introduce the vector $k_1^2=-1$, such that $u_1\cdot k_1=0$ and $u_2=\gamma u_1+\sqrt{\gamma^2-1}k_1$. Since the integrals will only have infrared divergences, we take $\varepsilon<0$ (we refer to [22] for a discussion of the infrared divergences in classical gravity computations). The set of master integrals in (69) have three delta functions from the velocity cuts on the massive propagator as required for obtaining a classical piece of order $1/\hbar$ from the quantum amplitude [29]. It is the same class of post-Minkowskian master integrals as the one obtained by the word-line EFT approach of [49, 53, 130] .

对于 $u_1^2=u_2^2=1, \gamma=u_1\cdot u_2, q^2=-1$ 和 $u_1\cdot q=u_2\cdot q=0$ ，引入矢量 $k_1^2=-1$ 是很方便的，由此可得 $u_1\cdot k_1=0$ 和 $u_2=\gamma u_1+\sqrt{\gamma^2-1}k_1$ 。由于积分仅存在红外发散，我们取 $\varepsilon<0$ (关于经典引力计算中的红外发散讨论可参见文献 [22])。式 (69) 中的主积分集合从质量传播子的速度割处得到三个 δ 函数，这是从量子振幅得到 $1/\hbar$ 阶经典项所要求的。这类后闵可夫斯基主积分与 [49, 53, 130] 的世界线有效场论方法得到的主积分属于同一类。

Changing variables to the rapidity

将变量换为快度

$$\gamma=\frac{1}{2}\left(x+\frac{1}{x}\right), \tag{71}$$

the 40 master integrals satisfy the linear differential system

这 40 个主积分满足线性微分系统

$$\frac{d}{dx}\mathcal{I}(x,\varepsilon)=A(x,\varepsilon)\mathcal{I}(x,\varepsilon), \tag{72}$$

with (regular) singularities at $x=\{-1,+1\}$ corresponding to the static limit $\gamma^2=1$ and at $x=\{0,\infty\}$ corresponding to the high-energy regime $\gamma\to\infty$. There are as well apparent (Apparent singularities are singularities of $A(x,\varepsilon)$ where the general solutions $\mathcal{I}(x,\varepsilon)$ are holomorphic. The removal of the apparent singularities can be done using Libra [131] or using the ore_algebra package implementation in [132, 133] .) singularities at the roots of $x^4\left(416\varepsilon^3-354\varepsilon^2+91\varepsilon-6\right)+\ x^2\left(320\varepsilon^3-292\varepsilon^2+82\varepsilon-6\right)+416\varepsilon^3-354\varepsilon^2+91\varepsilon-6=0$. At this perturbative order, an elliptic sector arises with solutions that are products of elliptic functions [27, 53] .

在 $x=\{-1,+1\}$ 处存在(正则)奇点,对应静态极限 $\gamma^2=1$;在 $x=\{0,\infty\}$ 处存在(正则)奇点,对应高能区域 $\gamma\to\infty$。在 $x^4\left(416\varepsilon^3-354\varepsilon^2+91\varepsilon-6\right)+x^2\left(320\varepsilon^3-292\varepsilon^2+82\varepsilon-6\right)+416\varepsilon^3-354\varepsilon^2+91\varepsilon-6=0$ 的根处也存在表观奇点(表观奇点指 $A(x,\varepsilon)$ 的奇点,通解 $\mathcal{I}(x,\varepsilon)$ 在该处解析。可借助 Libra[131] 或 [132, 133] 中实现的 ore_algebra 包消去表观奇点。)。在该微扰阶下,出现椭圆分支,其解为椭圆函数的乘积 [27, 53] 。

Using the basis of master integrals as in the probe computation in [29], we can expand all the 40 master integrals in the potential region, giving boundary data. It is enough to obtain the complete solution of the differential system in terms of only planar integrals. By matching the post-Newtonian expansion, we find that the system is decomposable into six sectors depending on the regime of the loop momenta l_2 and l_3 of the master integrals in (69). There are three regions $I^i_{\rm PP}(\varepsilon)$ where the loop momenta l_2 and l_3 correspond to potential mode gravitons, two regions $I^i_{\rm PR}(\varepsilon)$ where loop momentum l_2 is in potential mode and loop l_3 momentum is in radiation mode, and a region $I_{\rm RR}(\varepsilon)$ where both loop momenta l_2 and l_3 correspond to gravitons in radiation mode.

沿用文献 [29] 探测计算中所用的主积分基,我们可以在势区域对全部 40 个主积分做展开,得到边界数据。这足以仅用平面积分得到微分系统的完全解。通过匹配后牛顿展开,我们发现该系统可按式 (69) 中主积分圈动量 l_2 和 l_3 的区域分解为六个分支。其中存在三个区域 $I^i_{\rm PP}(\varepsilon)$: 圈动量 l_2 和 l_3 对应势模引力子;两个区域 $I^i_{\rm PR}(\varepsilon)$: 圈动量 l_2 处于势模,圈动量 l_3 处于辐射模;还有一个区域 $I_{\rm RR}(\varepsilon)$: 两个圈动量 l_2 和 l_3 都对应辐射模引力子。

The piece of the amplitude proportional to the mass dependence $m_1^5m_2^2$ and $m_1^2m_2^5$ is the probe contribution given by the first four-graviton cut in Fig. 7. The expression was derived in Eqs. (5.19) and (5.18) of [29] with the result

振幅中正比于质量依赖关系 $m_1^5m_2^2$ 和 $m_1^2m_2^5$ 的部分是探针贡献,由图 7 中的第一个四引力子切割给出。该表达式已在文献 [29] 的式 (5.19) 和 (5.18) 中推导得到,结果为

$$
\begin{aligned}
\mathcal{M}_{\rm 4PM}^{\rm probe}\left(\gamma,\underline{q}^2\right) &= \lim_{\varepsilon\to 0}\frac{(8\pi G_N)^4}{\left|\underline{q}\right|^{-1+3\varepsilon}}m_1^2m_2^2\left(m_1^3+m_2^3\right)\frac{(1-2\varepsilon)^3}{(2-2\varepsilon)^4}\frac{c_3(\gamma,\varepsilon)}{(\gamma^2-1)^3}I^1_{\rm PP}(1,\varepsilon),\\
&= G_N^4\left(m_1^3+m_2^3\right)m_1^2m_2^2\left|\underline{q}\right|\pi^3\frac{35i\left(33\gamma^4-18\gamma^2+1\right)}{8\left(\gamma^2-1\right)},
\end{aligned}
\tag{73}
$$

with the coefficient $c_3(\gamma,\varepsilon)$ evaluated in Eq. (5.17) of [29], and the master integral $I^1_{\rm PP}(\varepsilon)$ is the massless sunset integral evaluated in Eq. (2.31) of [134]. It is the probe contribution denoted by $\mathcal{M}_4^p(\gamma)$ in [27].

其中系数 $c_3(\gamma,\varepsilon)$ 已在文献 [29] 的式 (5.17) 中计算得到,主积分 $I^1_{\rm PP}(\varepsilon)$ 是无质量日落积分,已在文献 [134] 的式 (2.31) 中计算得到。它就是文献 [27] 中记为 $\mathcal{M}_4^p(\gamma)$ 的探针贡献。

The next-to-probe contribution to the radial action $\mathcal{N}_{4\mathrm{PM}}^{\text{next-to-probe}}(\gamma)$ is exacted from the mass contributions $m_1^4 m_2^3$ or $m_1^3 m_2^4$ of the three-loop amplitude, arising from the graphs with one velocity cut on the left and two velocity cuts on the right in Fig. 7

径向作用量 $\mathcal{N}_{4\mathrm{PM}}^{\text{next-to-probe}}(\gamma)$ 的次探针贡献从三圈振幅的质量项 $m_1^4 m_2^3$ 或 $m_1^3 m_2^4$ 中提取，来源于图 7 中左侧一个速度切割、右侧两个速度切割的图

$$\mathcal{M}_{4\mathrm{PM}}^{\text{next}-\text{to-probe}}\left(\gamma, \underline{q}^2\right) = -3\mathcal{M}_{4\mathrm{PM}}^{\text{probe}}\left(\gamma, \underline{q}^2\right) + \lim_{\varepsilon\to 0} \frac{G_N^4 m_1^3 m_2^3 (m_1+m_2)}{\left|\underline{q}\right|^{-1+3\varepsilon}} \widehat{\mathcal{M}}_3(\gamma,\varepsilon),$$

(74)

with the expansion in the basis of the static master integrals

展开在静态主积分的基下

$$\widehat{\mathcal{M}}_3(\gamma,\varepsilon) = \frac{\pi^2}{2} c_{\mathrm{PP}}^2(\gamma) I_{\mathrm{PP}}^2(\varepsilon) + \sum_{i=1,3} c_{\mathrm{PP}}^i(\gamma,\varepsilon) I_{\mathrm{PP}}^i(\varepsilon)$$
$$+\left(4\left(\gamma^2-1\right)\right)^{-2\varepsilon} c_{\mathrm{RR}}(\gamma,\varepsilon) I_{\mathrm{RR}}(\varepsilon)$$
$$+\left(4\left(\gamma^2-1\right)\right)^{-\varepsilon} \sum_{i=1}^{2} c_{\mathrm{PR}}^i(\gamma) I_{\mathrm{PR}}^i(\varepsilon). \tag{75}$$

The four-dimensional limit, $\varepsilon \to 0$, is infrared finite and splits into two contributions

四维极限 $\varepsilon \to 0$ 是红外有限的，可拆分为两项贡献

$$\lim_{\varepsilon\to 0} \widehat{\mathcal{M}}_3\left(\gamma, \underline{q}^2, \varepsilon\right) = \widehat{\mathcal{M}}_3^{\mathrm{PP-RR}}\left(\gamma, \underline{q}^2\right) + \widehat{\mathcal{M}}_3^{\mathrm{PR}}\left(\gamma, \underline{q}^2\right). \tag{76}$$

The contribution from the potential-potential and the radiation-reaction regions

势-势区域和辐射反应区域的贡献

$$\widehat{\mathcal{M}}_3^{\mathrm{PP-RR}}\left(\gamma, \underline{q}^2\right) = \lim_{\varepsilon\to 0}\left(\sum_{i=1,3} c_{\mathrm{PP}}^i(\gamma,\varepsilon) I_{\mathrm{PP}}^i(\varepsilon) + \frac{c_{\mathrm{RR}}(\gamma,\varepsilon)}{\left(4\left(\gamma^2-1\right)\right)^{2\varepsilon}} I_{\mathrm{RR}}(\varepsilon)\right) \tag{77}$$
$$+\frac{\pi^2}{2} c_{\mathrm{PP}}^2(\gamma) I_{\mathrm{PP}}^2(0)$$
$$= -16i\left(4\mathcal{M}_4^t(\gamma)\log\left(\frac{\sqrt{\gamma^2-1}}{2}\right) + \mathcal{M}_4^{\pi^2}(\gamma) + \mathcal{M}_4^{\mathrm{rem}}(\gamma) + 3\mathcal{M}_4^p(\gamma)\right),$$

with the tail contribution

加上尾部贡献

$$-64i\mathcal{M}_4^t(\gamma) = -\frac{16i\gamma\left(2\gamma^2-3\right)\left(35\gamma^4-30\gamma^2+11\right)}{\left(\gamma^2-1\right)^{3/2}}\operatorname{arccosh}(\gamma)$$

$$+32i\left(35\gamma^4+60\gamma^3-150\gamma^2+76\gamma-5\right)\log\left(\frac{1+\gamma}{2}\right).$$

(78)

The expression (77) from the potential-potential and the radiation-reaction regions reproduces the conservative part derived in [27]. The radiation-radiation sector cancels the infrared divergence from the potential-potential region leading to a finite classical result [27].

来自势-势区域和辐射反应区域的式 (77) 重现了文献 [27] 中推导的保守部分。辐射-辐射部分抵消了势-势区域的红外发散，得到有限的经典结果 [27]。

The contribution from the potential-radiation region reads

势-辐射区域的贡献为

$$\widehat{\mathcal{M}}_3^{\mathrm{PR}}(\gamma) = \lim_{\varepsilon\to 0}\left(4\left(\gamma^2-1\right)\right)^{-\varepsilon}\sum_{i=1}^{2} c_{\mathrm{PR}}^i(\gamma)\, I_{\mathrm{PR}}^i(\varepsilon)$$

$$= \frac{3\pi}{2}\left(5\gamma^2-1\right)\frac{2\gamma^2-1}{\sqrt{\gamma^2-1}}\left(\frac{\gamma^2}{\gamma^2-1}+\frac{2\left(2\gamma^2-3\right)\gamma\operatorname{arccosh}(\gamma)}{\left(\gamma^2-1\right)^{\frac{3}{2}}}-\frac{16}{3}\right)$$

$$-\pi\mathcal{M}_4^t\frac{d}{d\gamma}\left(\frac{2\gamma^2-1}{\sqrt{\gamma^2-1}}\right). \tag{79}$$

We refer to section ”Observables in General Relativity from Amplitudes” for a discussion of the scattering angle implication. The present result has to be amended by a proper account of the radiation from the momentum kick. This computation is not yet available in the scattering amplitude formalism, but a comparison to the world-line derivation of [54] gives a high-energy $\gamma \to \infty$ limit to the scattering amplitude that fails to reproduce the expected behavior of γ^2 dominated by the probe contribution according to (7.1) of [9]. All the results derived in the literature [27, 53, 54, 128] , at the time of writing this chapter, fail to reproduce this behavior. It is suggested by [135] that non-perturbative effects could tame this behavior. At one-loop order, a determination of the gravitational radiation and reaction has been using the KMOC formalism in [136] and using the heavy-particle effective theory in [137, 138] and scattering amplitudes in [139] .

关于散射角的相关讨论请参见章节“从振幅看广义相对论中的可观测量”。现有结果需要补充对动量反冲动量辐射的正确描述。该计算目前在散射振幅形式体系中尚未完成，但与文献 [54] 的世界线推导对比可知，散射振幅的高能 $\gamma \to \infty$ 极限无法重现文献 [9] 式 (7.1) 中由探针贡献主导的 γ^2 的预期行为。截至撰写本章，现有文献 [27, 53, 54, 128] 中所有结果都无法重现该行为。文献 [135] 指出非微扰效应可能修正该行为。单圈阶下，引力辐射和反作用已经在文献 [136] 中通过 KMOC 形式体系计算得到，在 [137, 138] 中通过重粒子有效理论计算得到，在 [139] 中通过散射振幅计算得到。

A complete understanding of the problem of the gravitational reaction at this order in perturbation is still an open question.

在该微扰阶下完全理解引力反作用问题仍然是一个开放问题。

Black Hole Metrics

黑洞度规

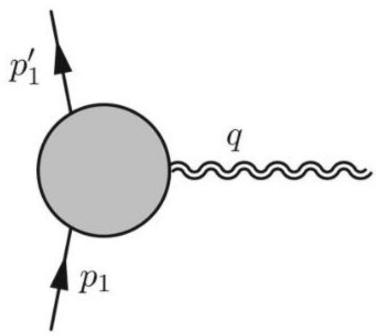
The Nobel Prize citation for Roger Penrose states that ”black hole formation is a robust prediction of the general theory of relativity.” Subrahmanyan Chandrasekhar explained that they are the most perfect macroscopic objects in the universe since the only elements in their construction are our concepts of space and time. Black hole solutions are an ideal playground to validate the formalism of deriving classical gravity from quantum scattering amplitudes. It also opens new avenues for studying black holes in generalized theories of gravity.

罗杰·彭罗斯的诺贝尔奖引证指出:“黑洞形成是广义相对论一个可靠的预言。”苏布拉马尼扬·钱德拉塞卡解释道，黑洞是宇宙中最完美的宏观物体，因为构成黑洞的唯一要素是我们对空间和时间的概念。黑洞解是验证从量子散射振幅推导经典引力这一形式体系的理想平台，也为研究广义引力理论中的黑洞开辟了新途径。

By evaluating the vertex function of the emission of a graviton from a particle of mass m , spin S , and charge Q , in d dimensions,

通过计算质量为 m 、自旋为 S 、电荷为 Q 的粒子在 d 维中发射引力子的顶点函数,

(80)



$$= -\frac{i\sqrt{32\pi G_N}}{2}\sum_{l\geq 0}\langle T^{(L)\mu\nu}(q^2)\rangle\varepsilon_{\mu\nu}\,.$$

At each loop order, we extract the L -loop contribution to the transition density of the stress-energy tensor $\langle T_{\mu\upsilon}(q^2)\rangle = \sum\limits_{l\geq 0}\left\langle T^{(L)}_{\mu\upsilon}(q^2)\right\rangle$

在每个圈阶，我们提取出应力能量张量 $\langle T_{\mu\upsilon}(q^2)\rangle = \sum\limits_{l\geq 0}\left\langle T^{(L)}_{\mu\upsilon}(q^2)\right\rangle$ 跃迁密度的 L 圈贡献

$$i\mathcal{M}_3^{(L)}(p_1,q) = -\frac{i\sqrt{32\pi G_N}}{2}\left\langle T^{(L)\mu\upsilon}(q^2)\right\rangle\varepsilon_{\mu\upsilon}, \tag{81}$$

where $\varepsilon^{\mu\upsilon}$ is the polarization of the graviton with momentum $q = p_1 - p_2$ as the momentum transfer. We also define $d \equiv D-1$. The de Donder gauge relation between the metric perturbation $g_{\mu\nu} = \eta_{\mu\nu} + \sum\limits_{n>1} h^{(n)}_{\mu\nu}$ and the stress-energy tensor reads

其中 $\varepsilon^{\mu\upsilon}$ 是动量为 $q = p_1 - p_2$ (即动量转移) 的引力子偏振。我们还定义了 $d \equiv D - 1$ 。度规微扰 $g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n>1} h_{\mu\nu}^{(n)}$ 与应力能量张量之间的德唐规范关系为

$$h_{\mu\nu}^{(L+1)}(\mathbf{x}) = -16\pi G_N \int \frac{d^d\mathbf{q}}{(2\pi)^d} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{1}{\mathbf{q}^2} \left(\left\langle T_{\mu\nu}^{(L)} \right\rangle^{\text{class}} \cdot (q^2) - \frac{1}{d-1} \eta_{\mu\nu} \left\langle T^{(L)} \right\rangle^{\text{class}} \cdot (q^2) \right). \tag{82}$$

In this relation enters the contribution at L -loop order $\left\langle T_{\mu\upsilon}^{(L)} \right\rangle^{\text{class.}} (q^2)$ defined by the classical limit of the quantum scattering amplitude [10,115,116].

该关系中包含了由量子散射振幅经典极限 [10,115,116] 定义的 L 圈阶贡献 $\left\langle T_{\mu\upsilon}^{(L)} \right\rangle^{\text{class.}} (q^2)$ 。

With the action

结合作用量

$$\mathcal{S} = \int d^{d+1}x\sqrt{-g}\left(\frac{R}{16\pi G_N} + \frac{1}{2}g^{\mu\upsilon}\partial_\mu\phi\partial_\upsilon\phi - \frac{1}{2}m^2\phi^2\right), \tag{83}$$

one can extract the metric of physical black holes:

我们可以推导出物理黑洞的度规:

- Schwarzschild black hole: Scalar field $S = 0$, mass m [6, 65, 134, 140] . In [134], the Schwarzschild metric up to G_N^4 was obtained in four $(d = 3)$, five $(d = 4)$, and six $(d = 5)$ dimensions.

- 史瓦西黑洞: 标量场 $S = 0$, 质量 m [6, 65, 134, 140] 。文献 [134] 在四维 $(d = 3)$ 、五维 $(d = 4)$ 和六维 $(d = 5)$ 中得到了最高到 G_N^4 阶的史瓦西度规。

- Reissner-Nordström black hole: Scalar field $S = 0$, charge Q , mass m [112].

- 赖斯纳-诺德斯特龙黑洞: 标量场 $S = 0$, 电荷 Q , 质量 m [112]。

- Kerr-Newman black hole: Fermionic field $S = \frac{1}{2}$, charge Q , mass m [65,112].

- 克尔-纽曼黑洞: 费米场 $S = \frac{1}{2}$, 电荷 Q , 质量 m [65,112]。

The derivation of the metric solutions in four dimensions from the scattering amplitude approach is a consistency check on the way one can embed classical Einstein's gravity into the standard massless spin-2 quantization of gravitational interactions. But, we can apply this to any effective field theory of gravity and for deriving a black hole metric. It is particularly interesting since we expect black holes to have exciting behavior in higher-dimensional theories of gravity [109, 141].

用散射振幅方法推导四维度规解，是对如何将经典爱因斯坦引力纳入引力相互作用的标准无质量自旋 2 量子化的一致性检验。但该方法可应用于任意引力有效场论，也可用于推导黑洞度规。由于我们预期黑洞在高维引力理论中会呈现有趣的性质 [109, 141]，因此该研究尤为有意思。

In the same way one can extract classical contributions from the amplitude computation, one can start processing quantum corrections in the amplitude in the effective field theory approach to gravity. We will not discuss this here since it requires a refined partial wave expansion to interpret such terms correctly. We will, however, emphasize the unique $\hbar$ -dependent contributions computed in [65,112] and that the effective field approach opens up possibilities for investigating further low-energy consequences of quantum contributions to scattering amplitudes.

正如我们可以从振幅计算中提取经典贡献，我们也可以在引力的有效场论框架下处理振幅中的量子修正。由于需要精细的分波展开才能正确解释这些项，本文在此不展开讨论。但我们需要强调文献 [65,112] 中计算得到的独特的依赖 $\hbar$ 的贡献，并且有效场方法为进一步研究量子贡献对散射振幅的低能效应开辟了新可能。

Observables in General Relativity from Amplitudes

从散射振幅得到广义相对论中的可观测量

Because, in general relativity, the gauge transformation is as well a coordinate frame change, the notion of observable differs from other quantum field theories, say electromagnetism, where one defines any gauge-invariant expression at a given time and position.

因为在广义相对论中，规范变换同时也是坐标变换，可观测量的概念与其他量子场论 (例如电磁学) 不同，后者是在给定时间和位置定义任意规范不变表达式。

In the scattering amplitude approach context, it is natural to connect to the scattering angle and the Shapiro time delay/advance, which are some of the fundamental classical observables in general relativity. The scattering angle is vital in connecting the various computations done in the post-Newtonian and post-Minkowskian calculations. The scattering angle is a central quantity for addressing the question of the radiation and the ultra-relativistic regime. One uses the Shapiro time delay to constrain various effective field theories of gravity [142-145].

在散射振幅方法的框架下，我们自然可以将其与散射角、夏皮罗时间延迟/提前联系起来，这二者都是广义相对论中基础的经典观测量。散射角对连接后牛顿近似和后闵可夫斯基近似中的各类计算至关重要，也是研究辐射问题和极端相对论区域的核心物理量。人们利用夏皮罗时间延迟约束引力的各类有效场论 [142-145]。

By applying the same framework to the gravitational scattering of a massless particle against a massive source, one can derive the second post-Minkowskian correction to the bending of light [82]

将相同框架应用于无质量粒子与大质量源的引力散射，即可推导出光偏折的二阶后闵可夫斯基修正 [82]

$$\theta \simeq \frac{4G_N M}{b} + \frac{15}{4}\frac{G_N^2 M^2 \pi}{b^2} + \dots \tag{84}$$

We see that the eikonal approximation leads to the expected classical general relativity contributions, in agreement with the next-to-leading correction of [146] and [147]. The contribution from John Donoghue in this volume [58] discusses the implication in the framework of an effective field theory treatment of quantum gravity.

我们可以看到，程函近似给出了符合预期的经典广义相对论贡献，与文献 [146] 和 [147] 的次领头阶修正结果一致。本卷中 John Donoghue 撰写的文章 [58] 在量子引力有效场论处理框架下讨论了这一结论的启示。

There are other observables like the impulse, which measures the net change in the momentum of one of the scattered particles in the initial state, which one can use to compute the classical interaction potential and the eikonal phase [116, 148].

还有其他可观测量，比如冲量，它测量初态中某一散射粒子动量的净变化，可用于计算经典相互作用势和程函相位 [116, 148]。

By scattering two massive spinless scalars, one can the scattering angle at the first and second post-Minkowskian order

通过散射两个有质量无标量粒子，可以得到后闵可夫斯基一阶和二阶的散射角

$$\chi_{1\text{PM}} = \frac{2\left(2\gamma^2-1\right)}{\sqrt{\gamma^2-1}}\frac{G_N m_1 m_2}{J},$$

$$\chi_{2\text{PM}} = \frac{3\pi}{4}\frac{m_1+m_2}{\mathcal{E}_{\text{C.M.}}}\left(5\gamma^2-1\right)\left(\frac{G_N m_1 m_2}{J}\right)^2, \tag{85}$$

where $\mathcal{E}_{\text{C.M.}}^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma$ and J is the angular momentum

其中 $\mathcal{E}_{\text{C.M.}}^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma$ 和 J 为角动量

$$J = \frac{m_1 m_2 \sqrt{\gamma^2-1}}{\mathcal{E}_{\text{C.M.}}} b \cos\left(\frac{\chi}{2}\right). \tag{86}$$

To this order, the scattering angle is the same as the one for the scattering of a test particle of mass $m_1 m_2/(m_1+m_2)$ in a static Schwarzschild background of mass m_1+m_2 .

在此阶下，散射角与质量为 $m_1 m_2/(m_1+m_2)$ 的试验粒子在质量为 m_1+m_2 的静态史瓦西背景中的散射角一致。

At the third post-Minkowskian order, the result gets more interesting as the result deviated from the one of a test particle in a Schwarzschild background and gravitational radiation enter a new and interesting way. At the third post-Minkowskian order, the scattering angle reads

在第三次后闵可夫斯基阶，结果变得更有趣，因为该结果偏离了史瓦西背景下测试粒子的结果，且引力辐射以一种新颖有趣的方式参与进来。在第三次后闵可夫斯基阶，散射角可表示为

$$\chi_{3\text{PM}} = \frac{2\left(64\gamma^6-120\gamma^4+60\gamma^2-5\right)}{3(\gamma^2-1)^{\frac{3}{2}}}\left(\frac{G_N m_1 m_2}{J}\right)^3$$

$$+\frac{8m_1m_2\sqrt{\gamma^2-1}}{3\mathcal{E}^2_{\text{C.M.}}}\left(-\gamma\left(25+14\gamma^2\right)+\frac{3\left(3+12\gamma^2-4\gamma^4\right)\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}\right)\left(\frac{G_Nm_1m_2}{J}\right)^3$$

$$+\frac{1}{\left(4\left(\gamma^2-1\right)\right)^{\frac{4-D}{2}}}\left(-\frac{11}{3}+\frac{d}{d\gamma}\left(\frac{\left(2\gamma^2-1\right)\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}\right)\right)$$

$$\times\frac{4m_1m_2\left(2\gamma^2-1\right)^2}{\mathcal{E}^2_{\text{C.M.}}}\left(\frac{G_Mm_1m_2}{J}\right)^3+\mathcal{O}(4-D). \tag{87}$$

The term multiplying $\left(4\left(\gamma^2-1\right)\right)^{\frac{4-D}{2}}$ is what can be viewed as radiation-reaction terms in [18, 19, 21, 23] . Because of infrared singularities, we regularize the scattering amplitude dimensionally at intermediate stages, but the final classical result is infrared finite in the limit $D\to 4$. One can derive radiation-reaction contributions using different amplitude-based methods: (1) high-energy scattering [17, 19], (2) linear response to the angular momentum [18, 149, 150] , and (3) reverse unitarity and the KMOC formalism [116, 151-153].

乘以 $\left(4\left(\gamma^2-1\right)\right)^{\frac{4-D}{2}}$ 的项可看作 [18, 19, 21, 23] 中的辐射反作用项。由于红外奇点，我们在中间步骤对散射振幅做维度正则化，但在 $D\to 4$ 极限下最终经典结果是红外有限的。人们可以用不同的基于振幅的方法推导辐射反作用贡献:(1) 高能散射 [17, 19]，(2) 角动量 [18, 149, 150] 的线性响应，(3) 逆幺正性与 KMOC 形式体系 [116, 151-153]。

At the next orders in the post-Minkowskian expansion, one can naïvely deduce a scattering angle from the amplitude contributions derived in section "Fourth-Order Post-Minkowskian Dynamics" by taking a Fourier transform with respect to the momentum transfer

在后闵氏展开的更高阶下，我们可以通过对“四阶后闵氏动力学”一节中得到的振幅贡献项做动量转移傅里叶变换，直接从中推导出散射角

$$\mathcal{M}_{4\text{PM}}(\gamma,|b|):=\int_{\mathbb{R}^2}e^{i\underline{q}\cdot b}\frac{\mathcal{M}_{4\text{PM}}\left(\gamma,\underline{q}^2\right)}{4m_1m_2\sqrt{\gamma^2-1}}\frac{d^2\underline{q}}{(2\pi)^2} \tag{88}$$

$$=-\frac{G_N^4m_1^2m_2^2(m_1+m_2)^3}{2\pi b^3}\left(c^{\text{probe}}(\gamma)+\frac{m_1m_2}{(m_1+m_2)^2}c^{\text{next-to-probe}}(\gamma)\right)$$

and determine the scattering angle by differentiating with respect to the center-of-mass angular momentum $\mathcal{J}_{\text{C.M.}}=m_1m_2\sqrt{\gamma^2-1}b/\mathcal{E}_{\text{C.M.}}$ to get

然后对质心系角动量 $\mathcal{J}_{\text{C.M.}}=m_1m_2\sqrt{\gamma^2-1}b/\mathcal{E}_{\text{C.M.}}$ 求导以确定散射角，得到

$$\chi_{4\text{PM}}(\gamma)=-\frac{\partial\mathcal{M}_{4\text{PM}}(\gamma,J)}{\partial\mathcal{M}_{\text{C.M.}}} \tag{89}$$

$$=\left(\frac{G_Nm_1m_2}{\mathcal{J}_{\text{C.M.}}}\right)^4\frac{\left(\gamma^2-1\right)(m_1+m_2)^3}{\mathcal{E}^3_{\text{C.M.}}}\left(c^{\text{probe}}(\gamma)+\frac{m_1m_2}{(m_1+m_2)^2}c^{\text{next-to-probe}}(\gamma)\right).$$

We decompose the scattering angle into a conservative part $\chi_{4\text{PM}}$ and corrections from the radiation-reaction

我们将散射角分解为保守部分 $\chi_{4\text{PM}}$ 和辐射反作用修正

$$\chi_{4\text{PM}} = \chi_{4\text{PM}}^{\text{cons}} + \delta^{\text{rr}} \chi_{4\text{PM}}^{\text{rel}}. \tag{90}$$

From the probe in (73) and potential-potential and radiation-radiation sectors of the next-to-probe piece of the amplitude in (77), we obtain the conservative part of the angle at the fourth post-Minkowskian order

结合式 (73) 中的探测项，以及式 (77) 中振幅次探测项部分的势-势 sector 与辐射-辐射 sector，我们得到了四阶后闵可夫斯基阶散射角的保守部分

$$\chi_{4\text{PM}}^{\text{cons}} = -\frac{3}{8\pi} \frac{(\gamma^2-1)(m_1+m_2)^3}{\mathcal{E}_{\text{C.M.}}^3} \left(\frac{G_N m_1 m_2}{\mathcal{J}_{\text{C.M.}}}\right)^4 \left(c^{\text{probe}}(\gamma) + \frac{m_1 m_2}{(m_1+m_2)^2} \widehat{\mathcal{M}}_3^{\text{PR}}(\gamma)\right),$$

(91)

in agreement with the results of [27, 53, 127, 128] .

与 [27, 53, 127, 128] 的结果一致。

From the potential-radiation sector in (79), we obtain, using the notations of [127],

根据 (79) 中的势辐射部分，采用文献 [127] 的记号，我们得到

$$\delta^{\text{rr}} \chi_{4\text{PM}}^{\text{rel}} = \left(\frac{G_N m_1 m_2}{\mathcal{J}_{\text{C.M.}}}\right)^4 \left(2\chi_2^{\text{cons}} J_2 - h E_3 \frac{d\chi_1^{\text{cons}}}{d\gamma}\right). \tag{92}$$

This expression reproduces Eq. (7.9) of [127] except for the J_3 piece which is affected by the radiation loss [127, 150]. The recoil has been included in the word-line derivation in [54]. The gravitational radiation changes the relationship between the amplitude and the scattering angle. It has been argued in [154] that the eikonal connection between the momentum q -space analysis and the b -space results has to be corrected to include the angular momentum change from the radiation. The consequences in scattering amplitude approaches, which we have presented, are still being researched at the time of writing.

该表达式重现了文献 [127] 的式 (7.9)，仅 J_3 项不同，该项受辐射损失影响 [127, 150]。文献 [54] 的世界线推导中已经包含了反冲效应。引力辐射会改变振幅与散射角之间的关系。文献 [154] 提出，动量 q 空间分析与 b 空间结果之间的程函连接需要修正，以纳入辐射带来的角动量变化。在撰写本文时，我们所介绍的散射振幅方法中的相关结论仍在研究中。

Conclusion

结论

We deliver in this chapter an account of several diverse and exciting developments that we expect will drive forward the field of gravitational physics in the future. An essential starting point for the analysis is using the framework of effective field theory for perturbative gravitational interactions. This chapter guides how to arrive at a low-energy quantum extension of general relativity and is an ideal starting point for most contemporary phenomenological explorations of gravitational interactions. It is also essential in precision data analysis, where we compute gravitational scattering amplitudes at high perturbative orders. One can demonstrate such computations to be beneficial as a complement to numerical general relativity in the inspiral region of a binary black hole merger at relativistic velocities. Thus, gravitational scattering amplitude technology is vital anywhere we need computational precision for an accurate analysis. We anticipate that advancement in this field will persist along the research summarized here. A current bottleneck for such computations arises in determining an ansatz for the scattering amplitude at high perturbative orders. We also envision further improvements in integrand reconstructions at high multiplicities of interaction gravitons. The current state of the art in computation is the fifth post-Minkowskian order, and we envision progress in the coming years.

> 本章我们阐述了多个不同且令人振奋的进展，我们认为这些进展将推动引力物理学领域的未来发展。本分析的核心起点是采用有效场论框架研究微扰引力相互作用。本章介绍了如何得到广义相对论的低能量子扩展，这是当代大多数引力相互作用唯象研究的理想起点。它在精度数据分析中也必不可少，我们在此计算高微扰阶的引力散射振幅。已经证明，这类计算可对相对论速度下双黑洞合并并合旋进阶段的数值广义相对论研究形成有益补充。因此，任何需要计算精度来完成准确分析的场景中，引力散射振幅技术都至关重要。我们预计该领域将沿着本文总结的研究方向持续发展。这类计算当前的瓶颈在于如何确定高微扰阶散射振幅的拟设。我们也期待高 multiplicity 引力子相互作用的被积函数重建能取得进一步进展。当前该领域的最高计算水平是五阶后闵可夫斯基近似，我们预计未来几年将取得新突破。

Another area of attraction is the computation of two-to-two scattering amplitude with spinning massive particles. Since black holes have non-trivial classical spins, the consequences of having a non-trivial spin can have important effects on the analysis. We can account for a spin in current numerical relativity computations, but an interesting idea is to employ perturbative amplitudes with spin to complement effects. Again, we find the inspiral region of mergers intriguing since precision is indispensable. Here, investigations are behind the state-of-the-art progress for high perturbative probes, but there are multiple attractive paths to track for further progress. Since it is outside the extent of this chapter, we depart from such deliberations and leave them to forthcoming studies.

> 另一个引人关注的领域是带质量旋转粒子的二对二散射振幅计算。由于黑洞具有非平凡经典自旋，非平凡自旋带来的影响会对分析产生重要作用。当前数值相对论计算已经可以处理自旋，一个有趣的思路是利用带自旋的微扰振幅对这些效应做补充。同样，我们发现合并过程的旋进阶段非常值得研究，因为精度在这里不可或缺。目前该方向的研究进展落后于高微扰探测领域的顶尖水平，但仍存在多个有前景的研究路径可以推进。由于相关讨论超出了本章范围，我们不再展开，留待后续研究探讨。

Furnished with the effective field theory framework for a low-energy theory of gravity, it is also interesting to examine the potential consequences of high-derivative terms in the effective Lagrangian. The higher-derivative terms in the Lagrangian signal depart from Einstein's gravity; however, since curvature is small in most instances, it is (very) challenging to realize areas where we would be capable of affirming deviation from Einstein's theory of relativity. Present bounds for verifiable cosmological deviations are huge. Nevertheless, since an effective field appears to be the natural framework for a plausible theory of quantum gravity, we should scrutinize any manifestation of the existence of higher-derivative terms in the gravitational action.

借助引力低能理论的有效场论框架，研究有效拉格朗日量中高导数项的潜在影响也十分有意义。拉格朗日量中的高导数项意味着偏离爱因斯坦引力；但由于大多数情况下曲率很小，我们很难找到能够证实偏离爱因斯坦相对论的场景。目前可观测宇宙学偏差的约束范围很大，尽管如此，由于有效场论是合理量子引力理论的自然框架，我们应当仔细检验引力作用量中存在高导数项的任何可能迹象。

Increased resolution of observed gravitational wave signals stimulates quests for new physics, opening a fascinating chapter of physics in the years to come. With our new theoretical laboratory for tests of low-energy phenomenological extensions of Einstein's theory of gravity, we envision providing potential for observation-driven astrophysical discoveries.

观测到的引力波信号分辨率不断提升，推动了对新物理的探索，即将开启物理学中令人神往的新篇章。借助我们用于检验爱因斯坦引力低能唯象扩展的全新理论实验室，我们有望推动由观测驱动的天体物理学新发现。

Cross-References

交叉引用

Quantum General Relativity and Effective Field Theory

量子广义相对论与有效场论

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